Computing at least one point per connected components of semi-algebraic sets defined by a single inequality

Edern Gillot

SIAM AG25 – Computational Real Algebraic Geometry

Joint work with Jérémy Berthomieu and Mohab Safey El Din

11 July 2025

Problem Description

Real polynomial systems
with **constraints**

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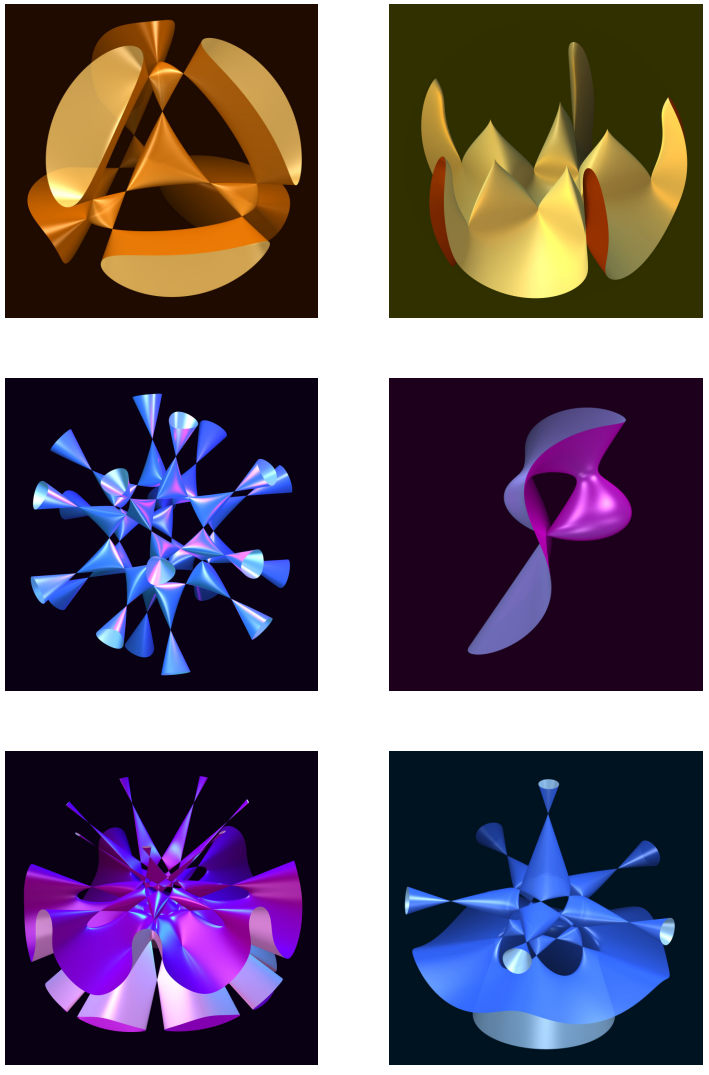

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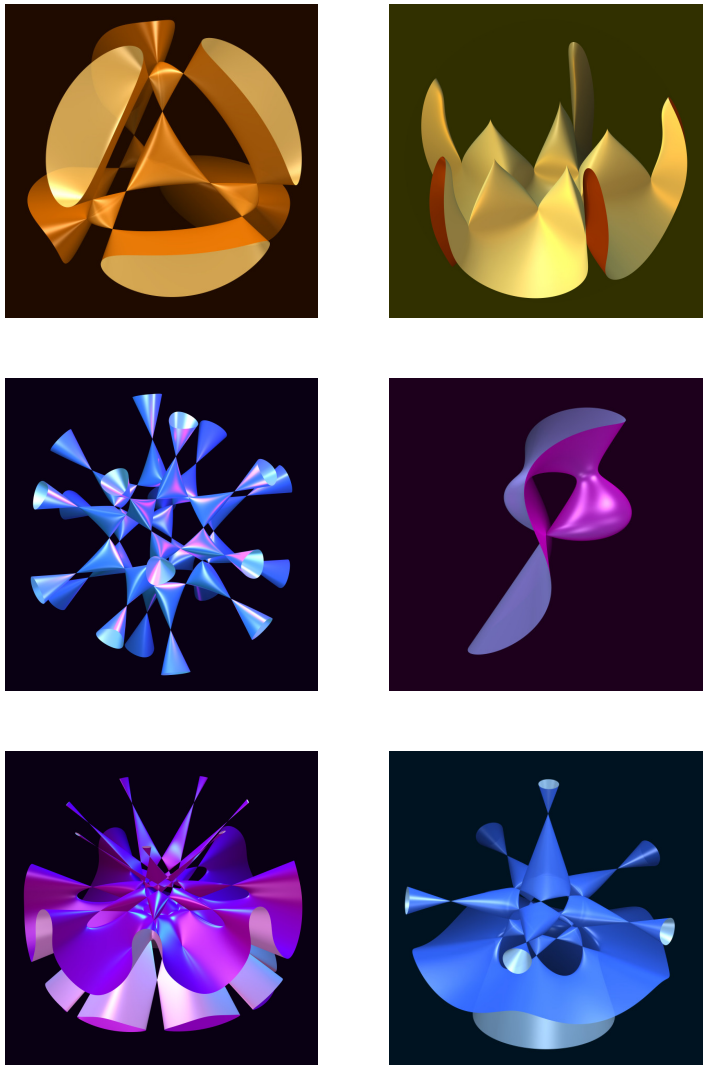

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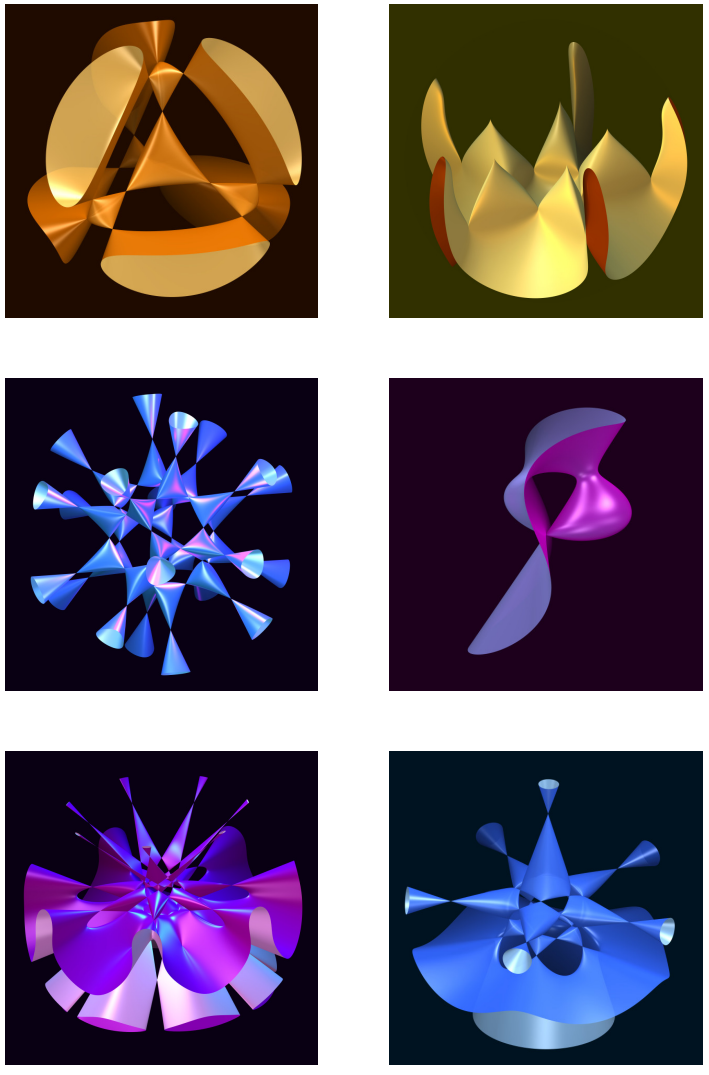

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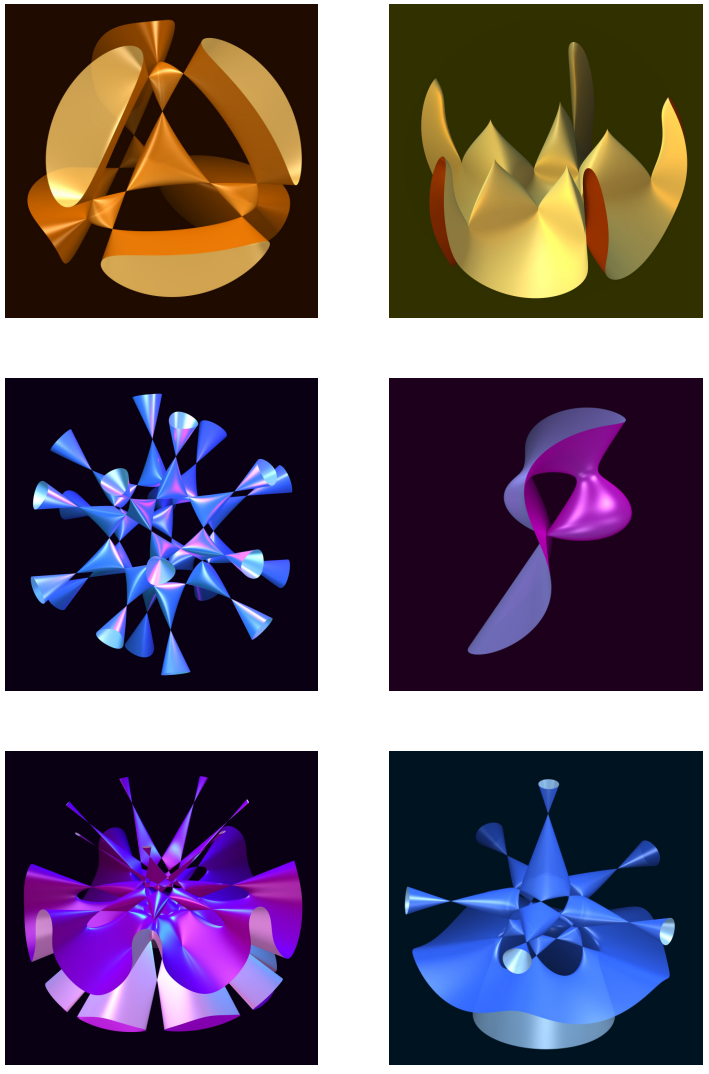

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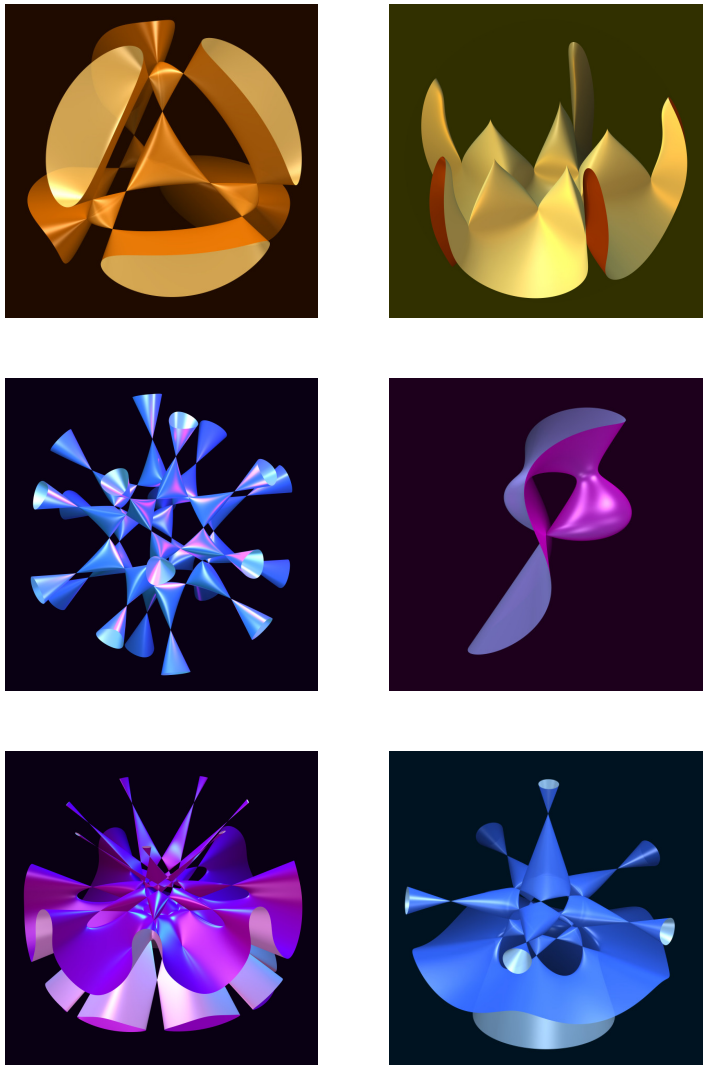

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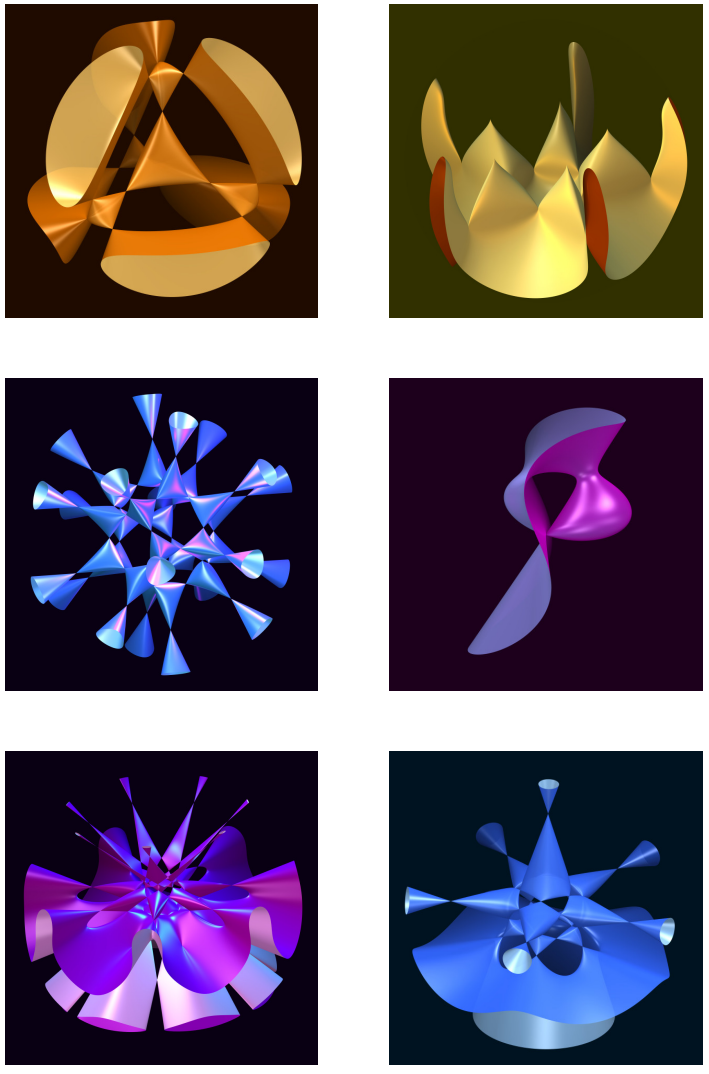

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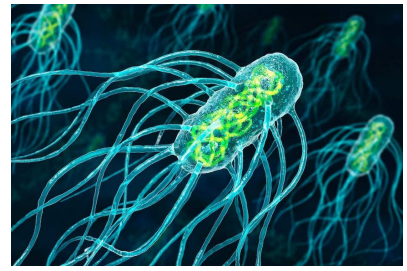

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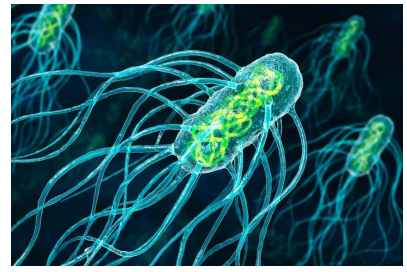

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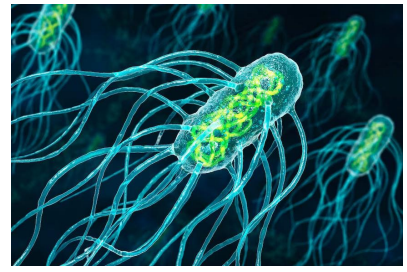

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Single inequality:

$$S = \{\boldsymbol{x} \in \mathbb{R}^n : f(\boldsymbol{x}) \neq 0\}$$

State-of-the-Art

n variables, degree d, bitsize τ, Thom–Milnor bound $O(d)^n$

First algorithm: [Tarski 1951] not elementary recursive

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Probabilistic algorithms

[Bank–Giusti–Heintz–Mandel–Mbakop 1997] [Safey El Din–Schost 2003]
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$$\widetilde{O}\Big(\mathcal{P}(n)d^2(d-1)^{2n}\Big)$$

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Algebraic to **semi-algebraic**
$f \neq 0 \to \lambda f - 1 = 0$

More variables & **breaks** structure

Contributions

Input: $f \in \mathbb{Q}[x_1, \ldots, x_n]$, $0 < \epsilon < 1$

Output: **parametrisation** of finite set of points with probability **at least** $1 - \epsilon$

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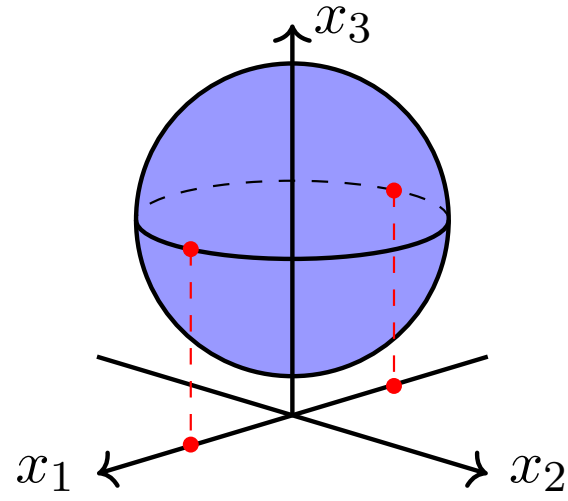

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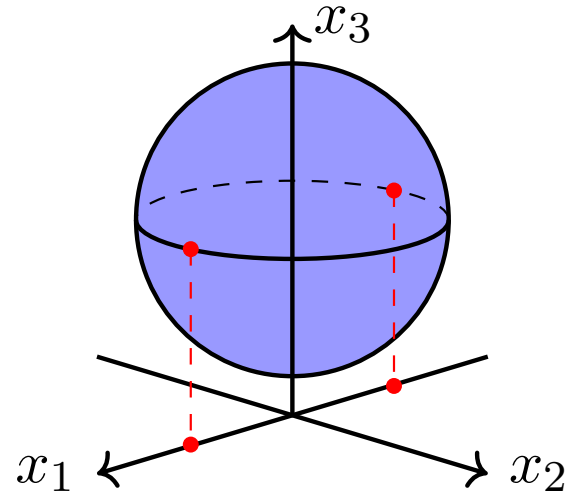
Critical points are defined by **partial derivatives**



Singular points **always** satisfy critical point equations

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Random change of variables

Properness of projection on x-axis

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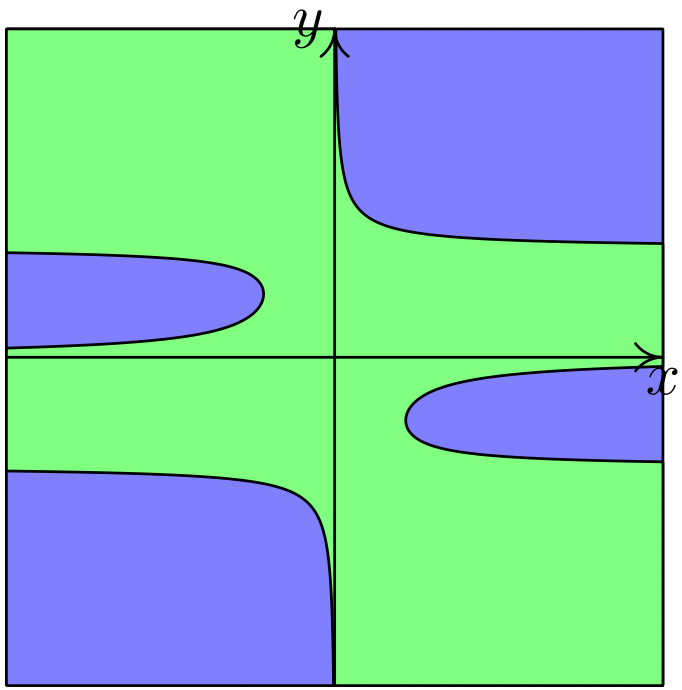

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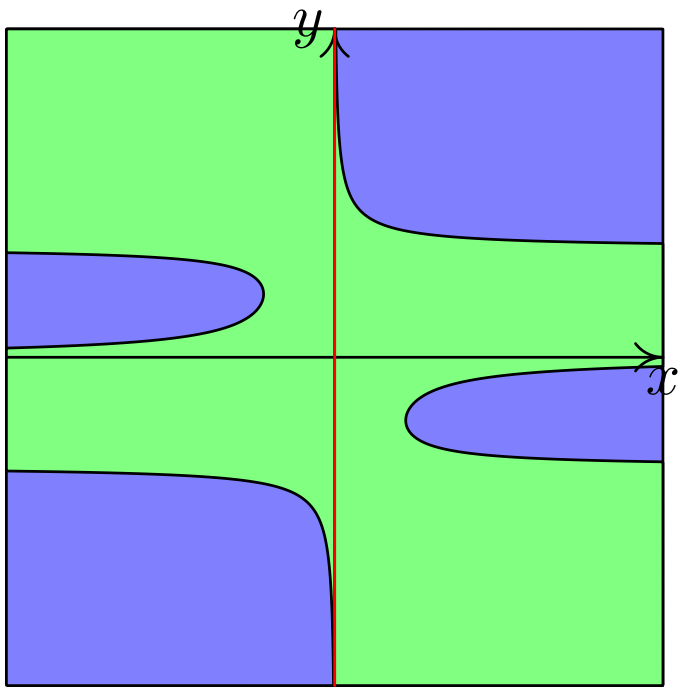

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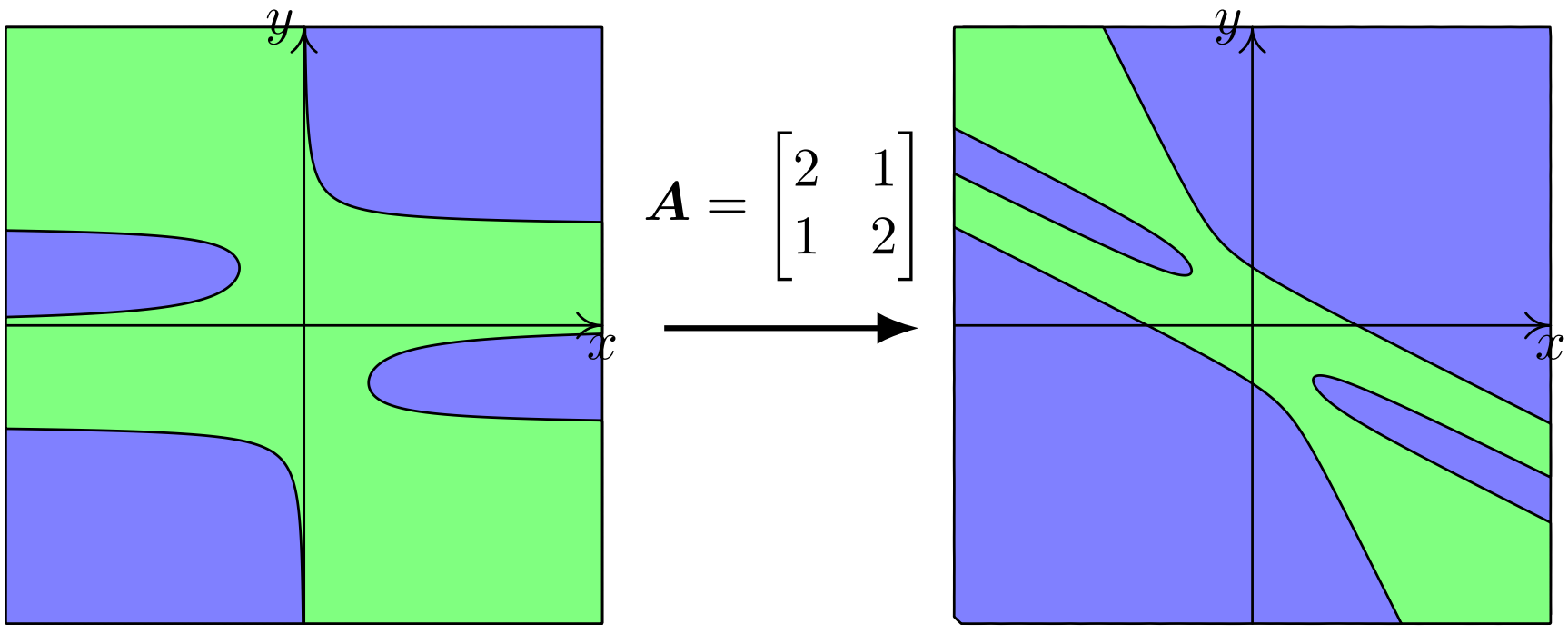

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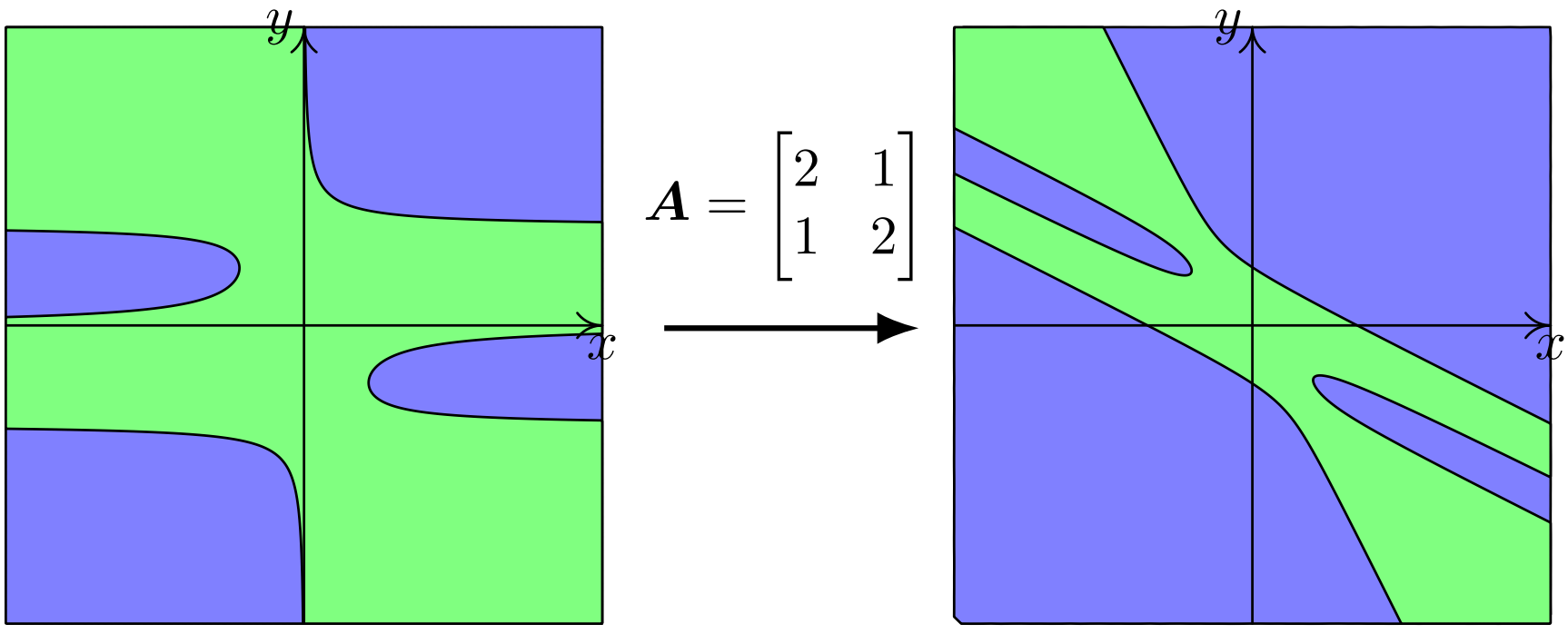

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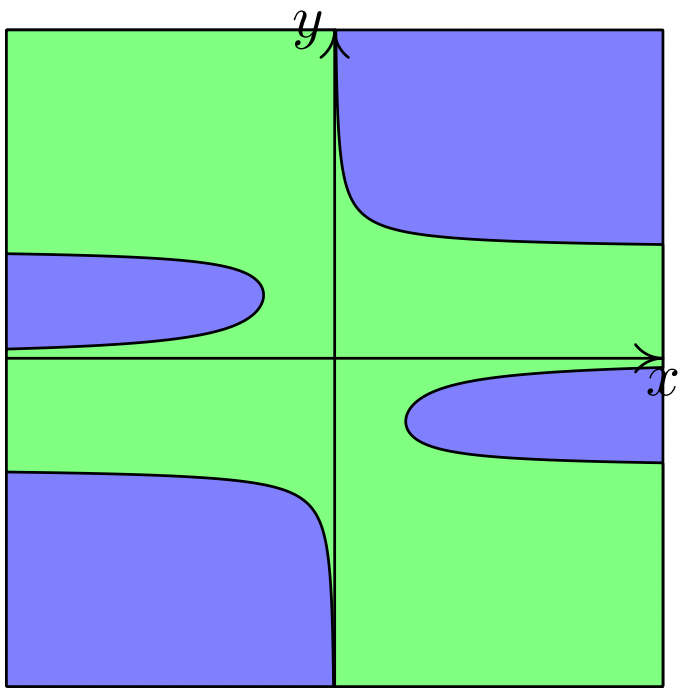

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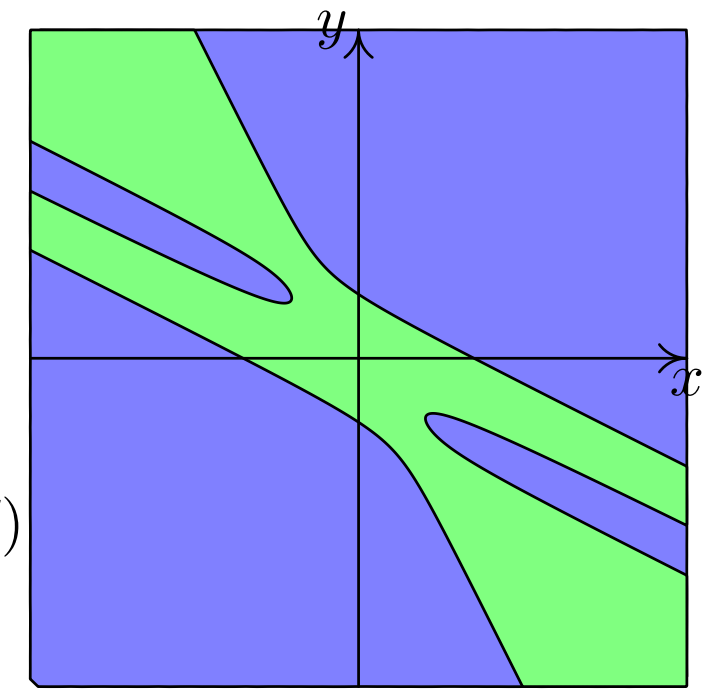

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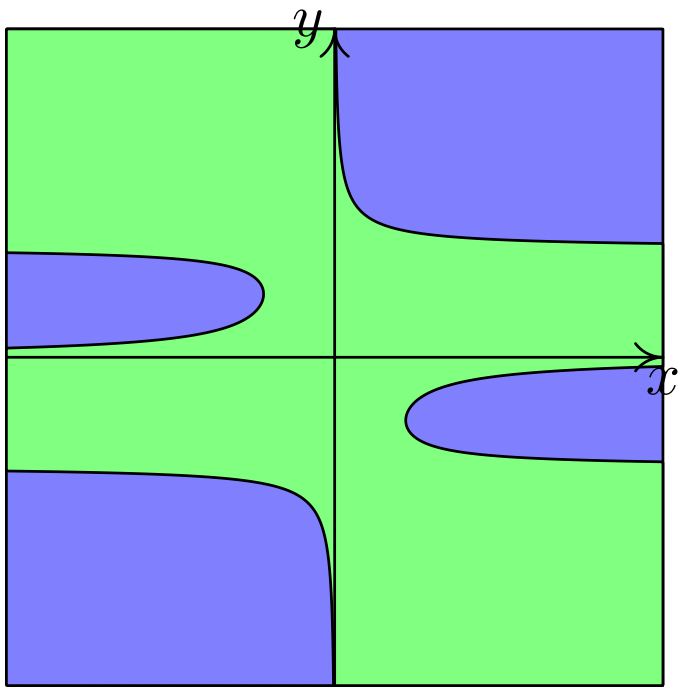

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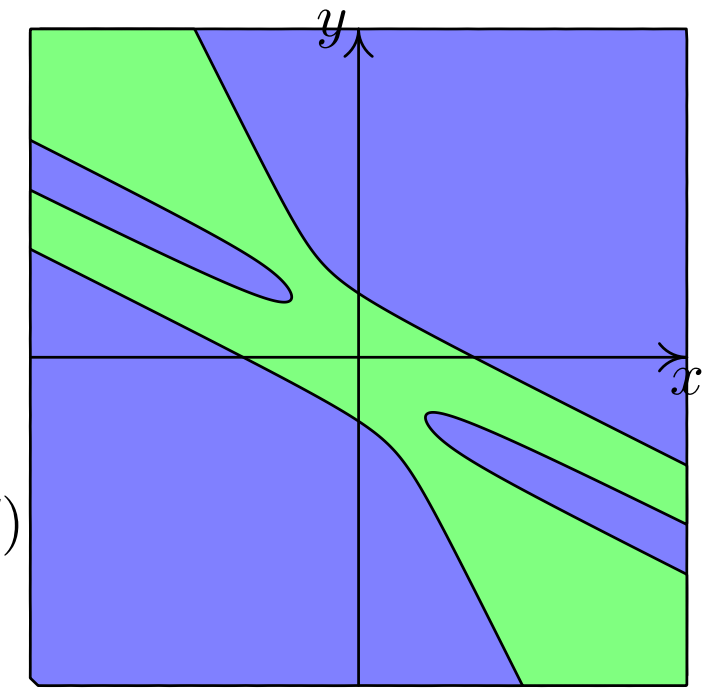

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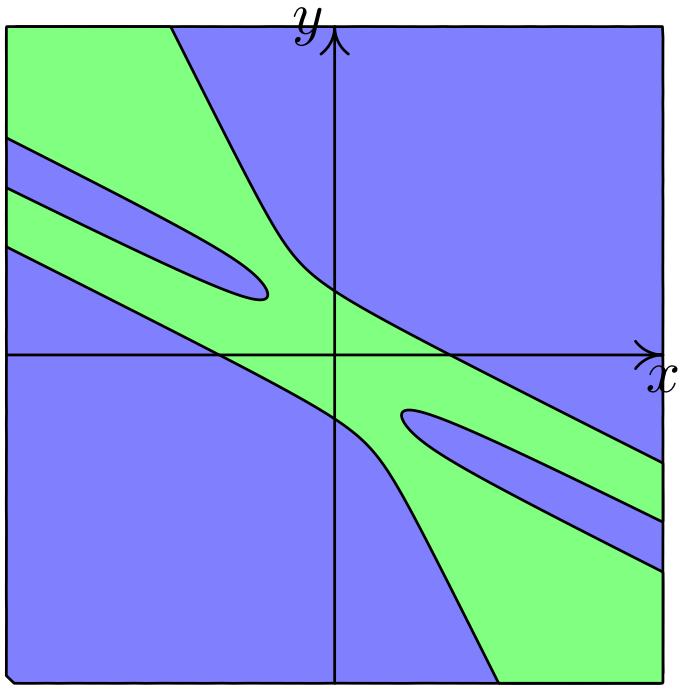

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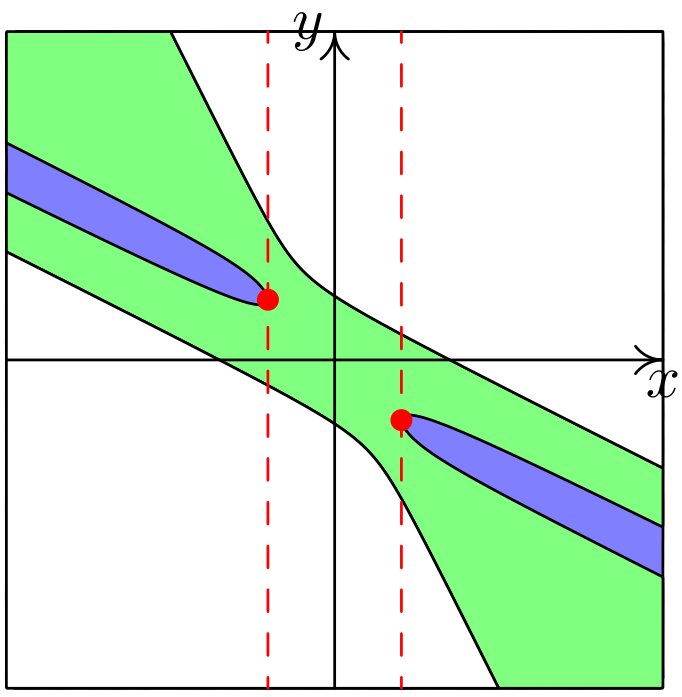

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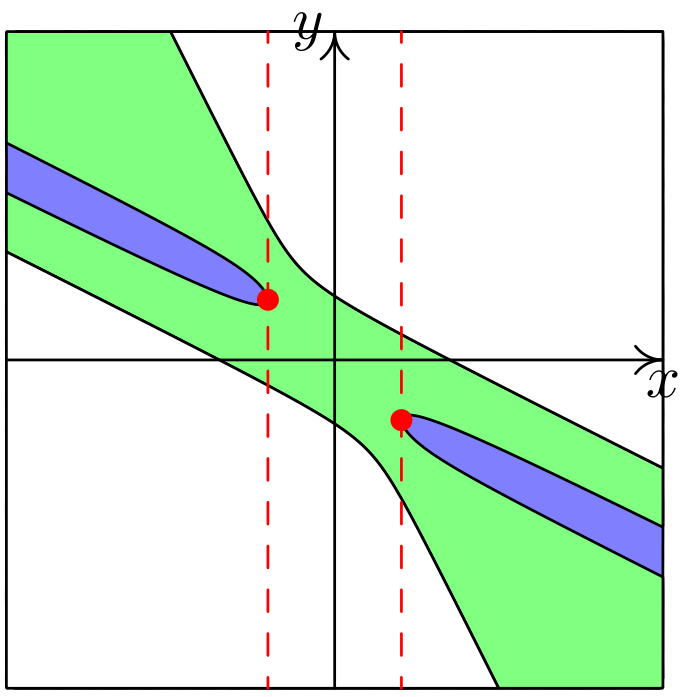

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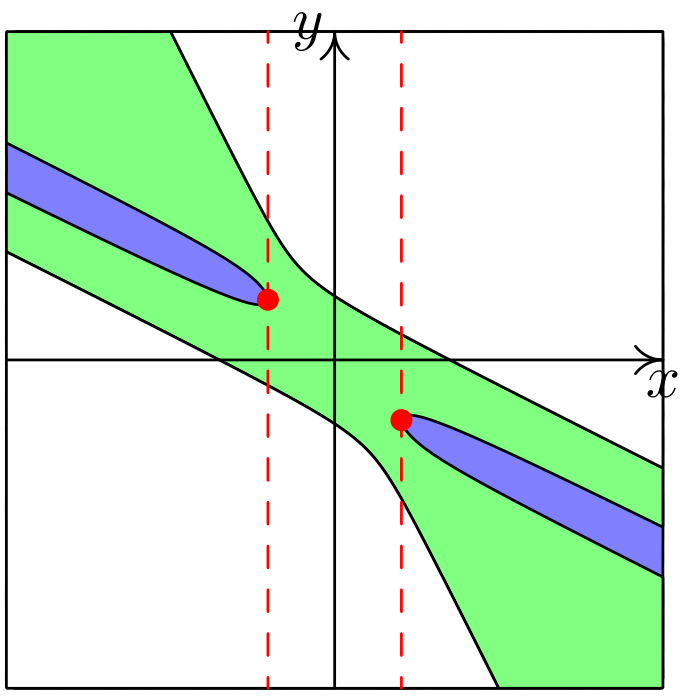
Bit cost for computing **parametrisation** of solutions:

quadratic in $\deg(f^{\boldsymbol{A}}) \prod_{i=2}^{n} \deg\left(\frac{\partial f^{\boldsymbol{A}}}{\partial X_i}\right)$

[Safey El Din–Schost 2018]

Smooth Case — Critical Points



$$f^{\boldsymbol{A}} = \frac{\partial f^{\boldsymbol{A}}}{\partial x_2} = \cdots = \frac{\partial f^{\boldsymbol{A}}}{\partial x_n} = 0$$

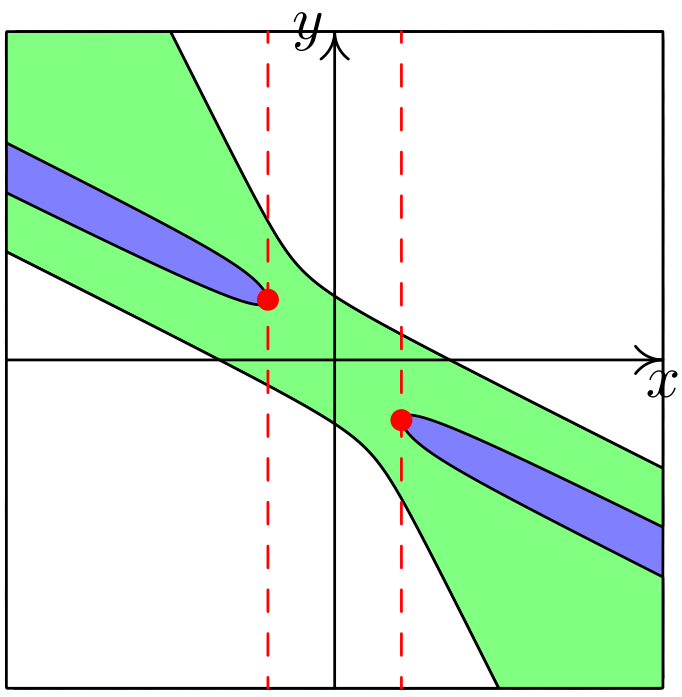
Bit cost for computing **parametrisation** of solutions:

quadratic in $\deg(f^{\boldsymbol{A}}) \prod_{i=2}^{n} \deg\left(\frac{\partial f^{\boldsymbol{A}}}{\partial X_i}\right)$

[Safey El Din–Schost 2018]

A breaks structure!

Smooth Case — Critical Points



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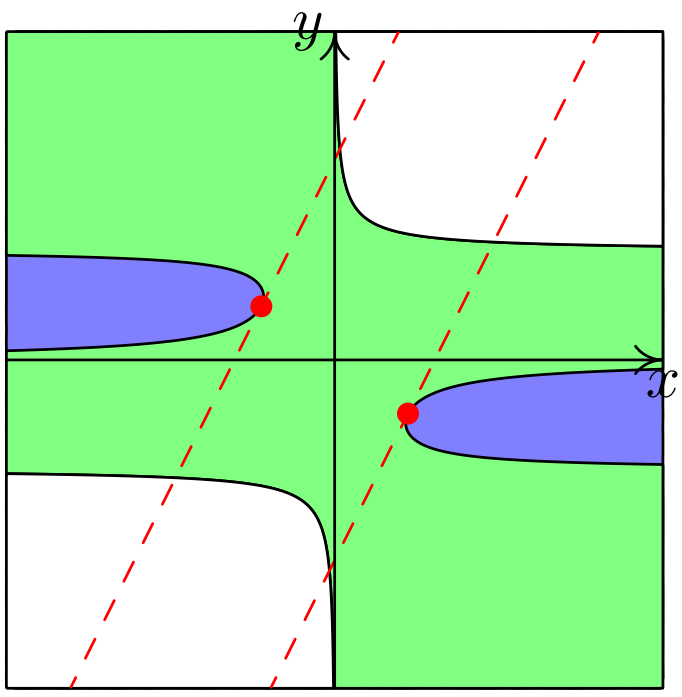
$\exists g_2, \ldots, g_n \in \mathbb{Q}[x_1, \ldots, x_n] : \deg(g_i) = \deg\left(\frac{\partial f}{\partial X_i}\right)$ such that

$\boldsymbol{\xi}$ critical $\iff$ $\boldsymbol{A\xi} \in V(f, g_2, \ldots, g_n)$

Smooth Case — Critical Points



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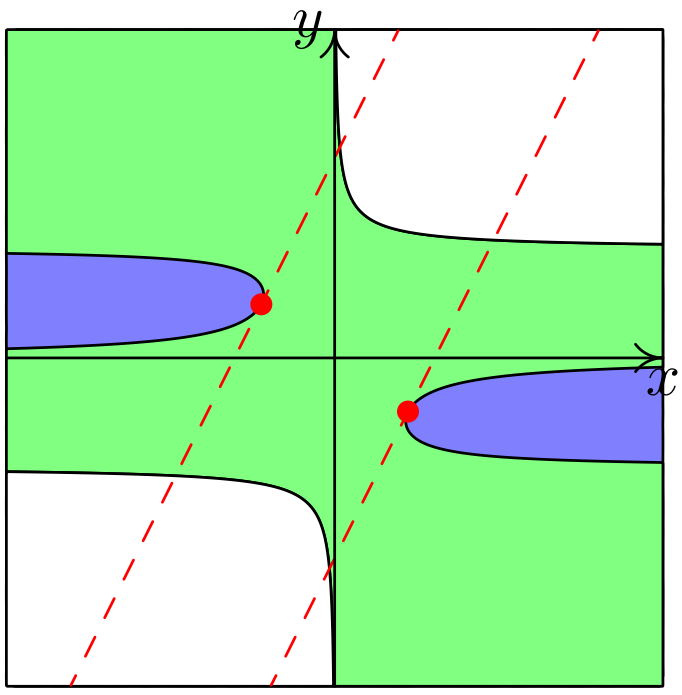

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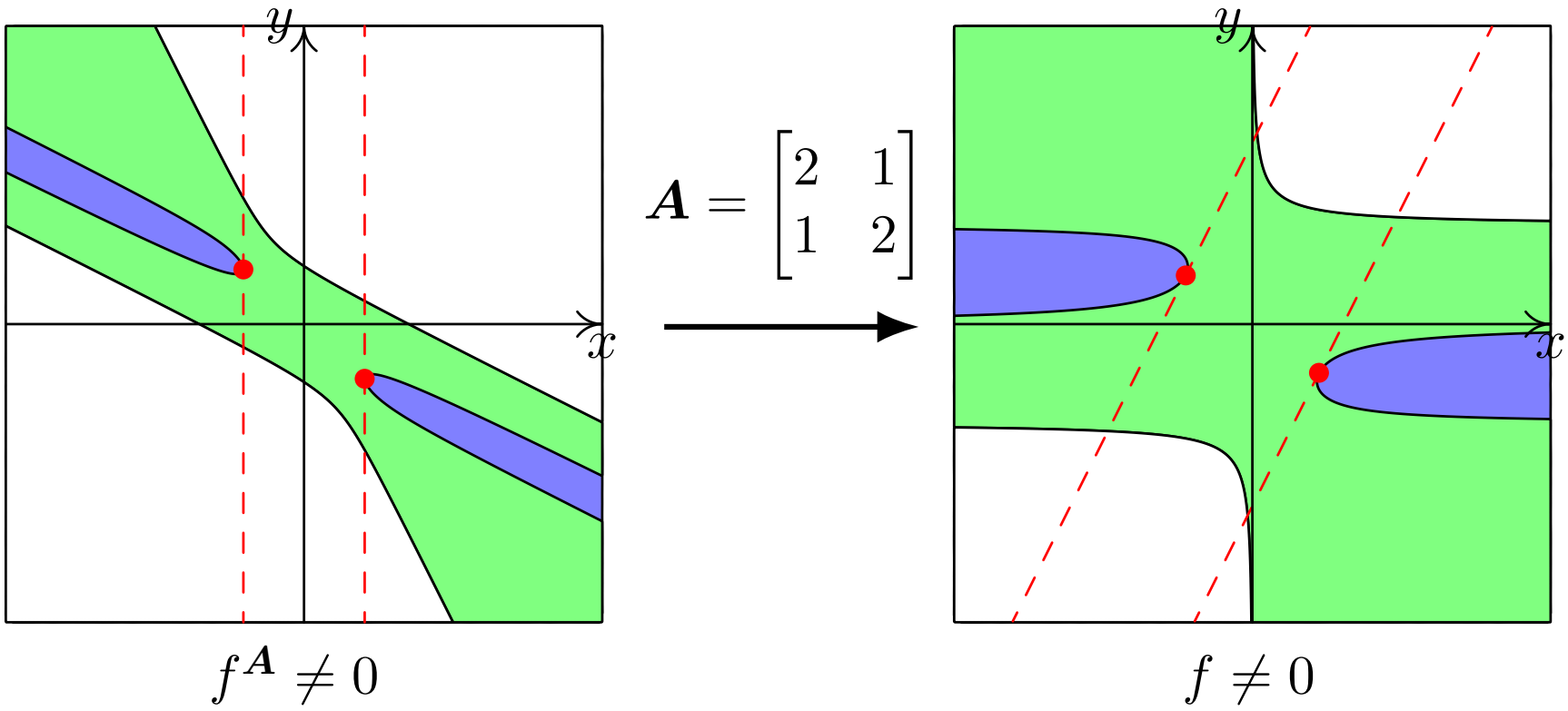
$\boldsymbol{\xi}$ critical $\iff$ $\boldsymbol{A\xi} \in V(f, g_2, \ldots, g_n)$

Bit cost for computing these points: $\widetilde{O}((\tau + \log(1/\epsilon))\mathcal{P}(n)\mathfrak{D}^2)$

Probability of success $\geq 1 - \epsilon/3$ after $O(\log(1/\epsilon))$ attempts

Smooth Case — Transverse Line



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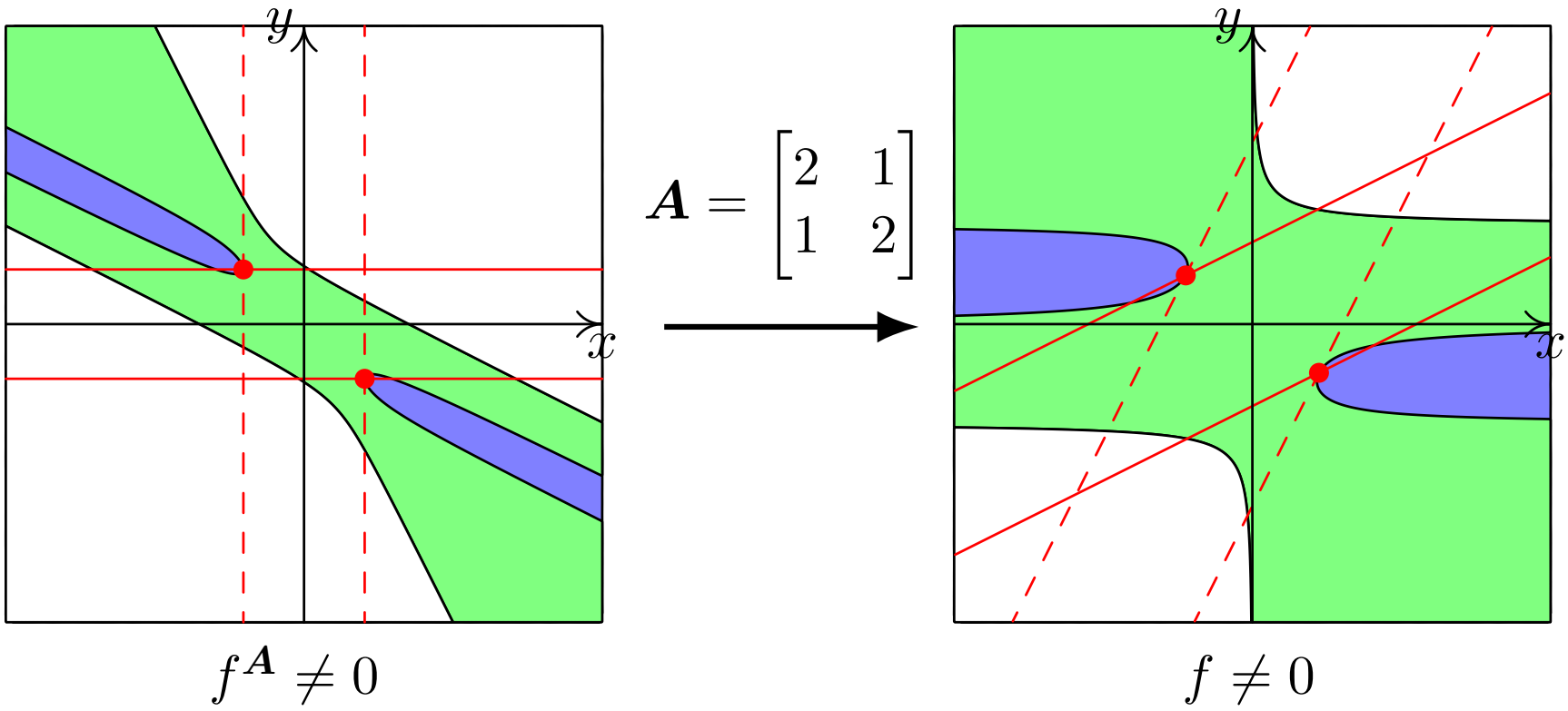

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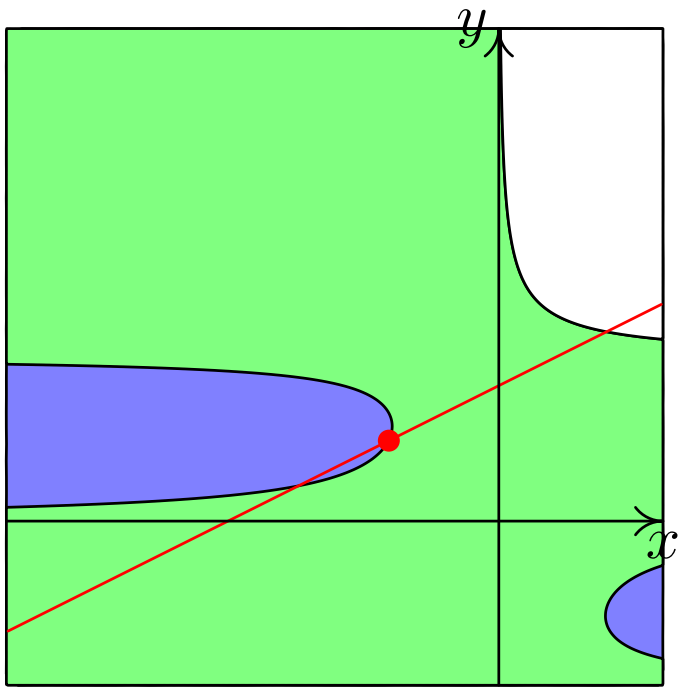
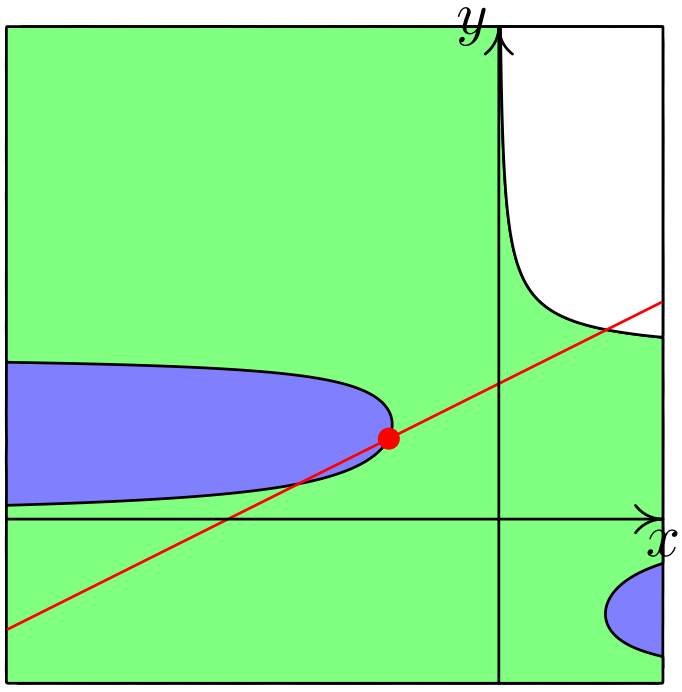


$f \neq 0$

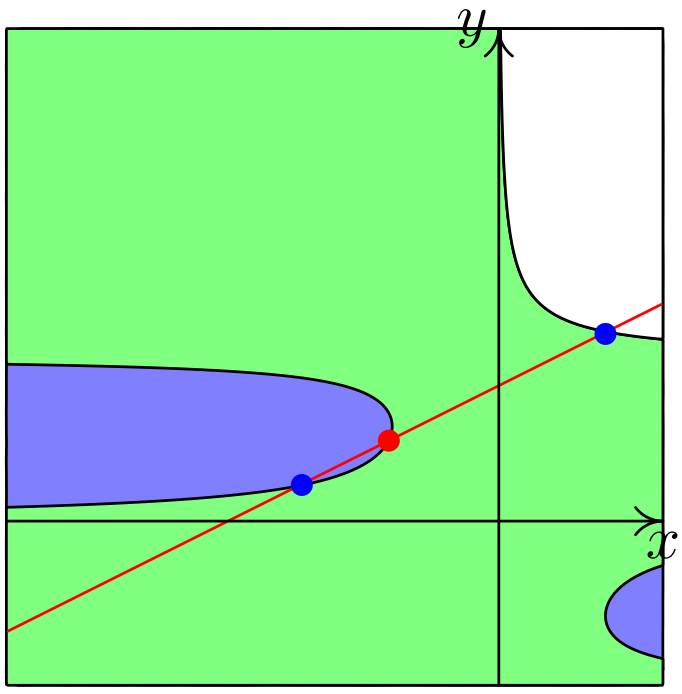
Smooth Case — Transverse Line



$f \neq 0$

Minimal distance between **computed solution point** and other **intersection points**

Smooth Case — Transverse Line



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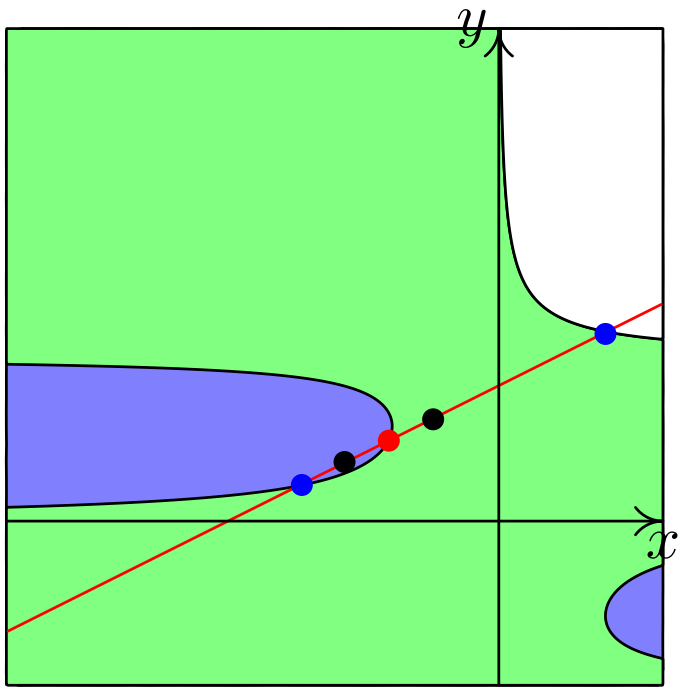

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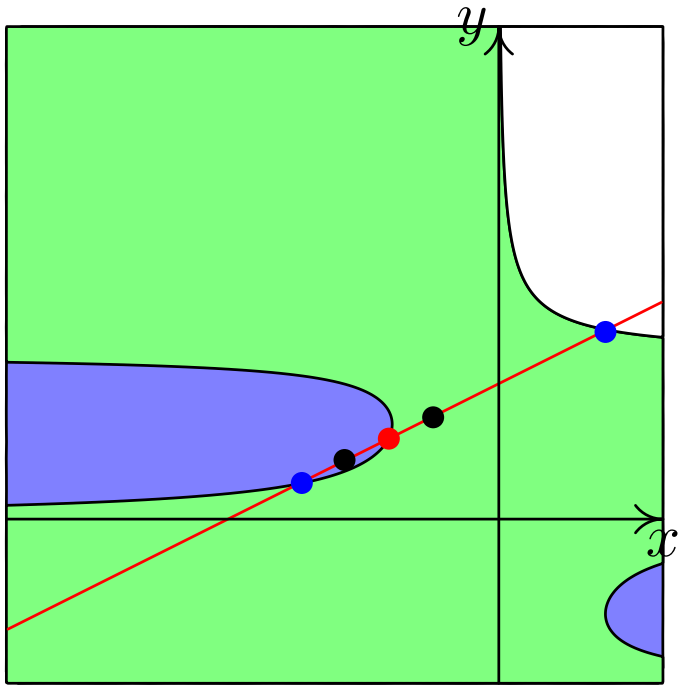


$f \neq 0$

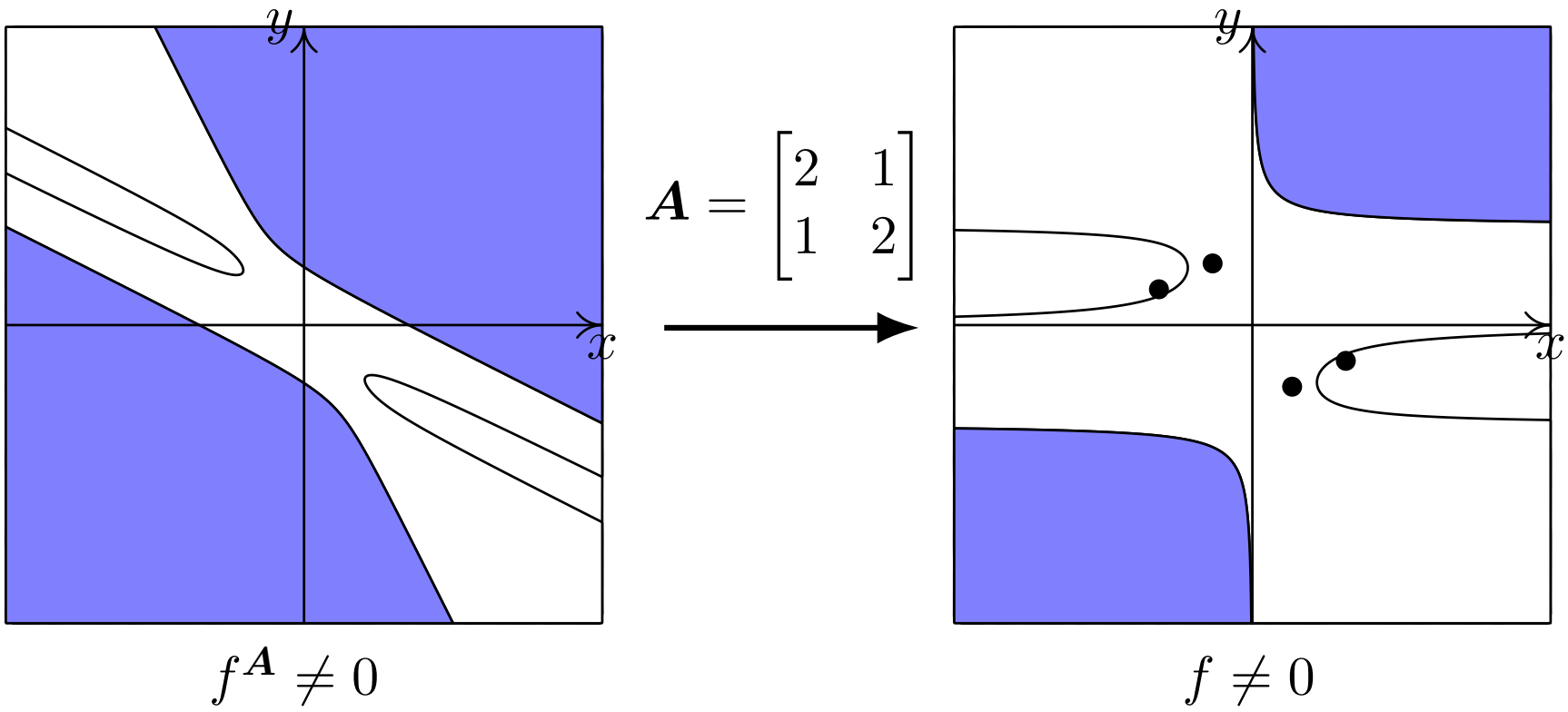
Minimal distance between **computed solution point** and other **intersection points**

Number of real critical points $\mathfrak{R} \leq \mathfrak{D}$

Bit cost of computing **minimal distance** and **parametrisations** of the **black points**: $\tilde{O}((\tau + \log(1/\epsilon))\mathcal{P}(n)\mathfrak{R}\mathfrak{D}^2)$

[Strzebonski–Tsigaridas 2019]

Smooth Case — Specification



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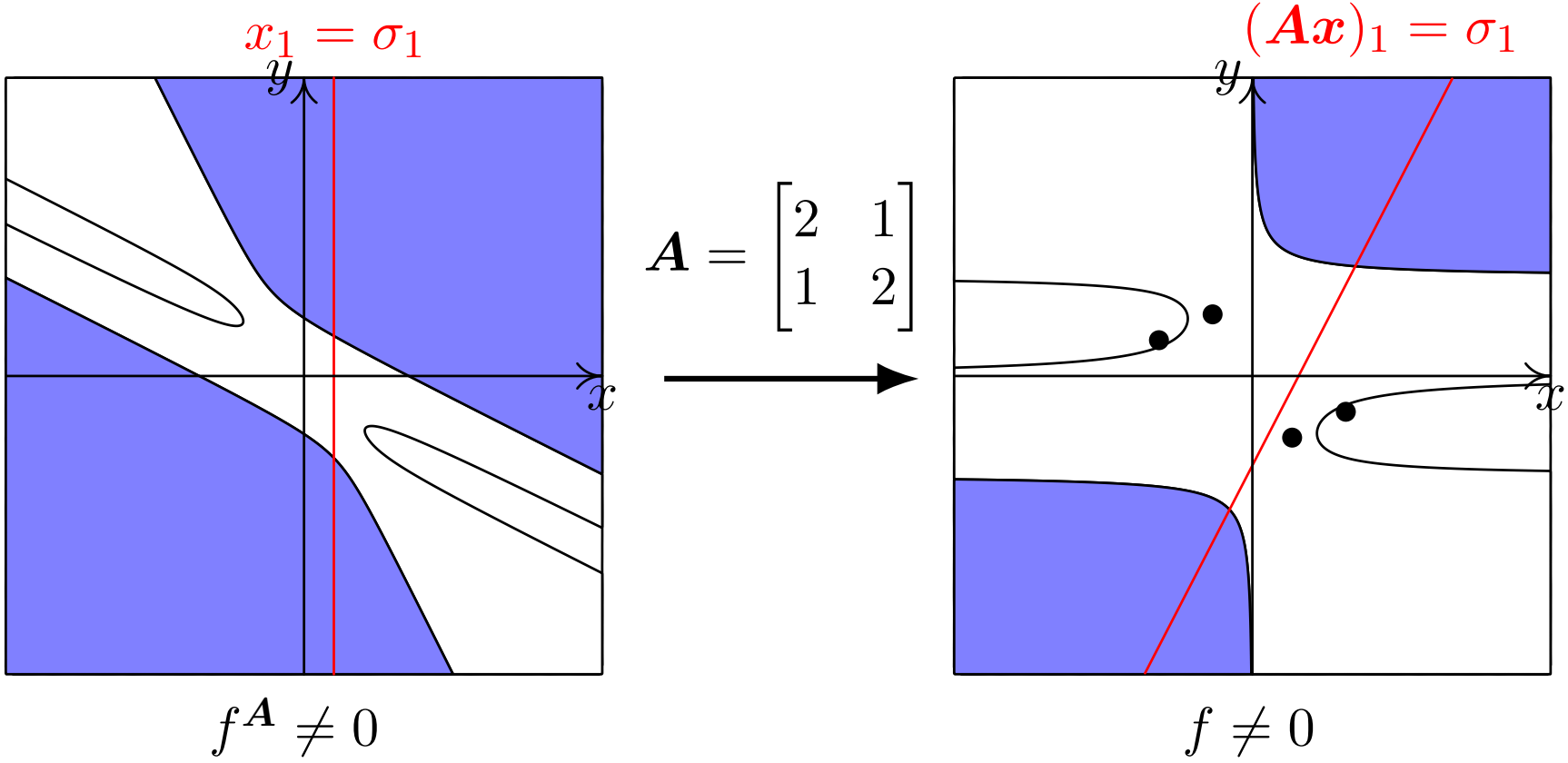

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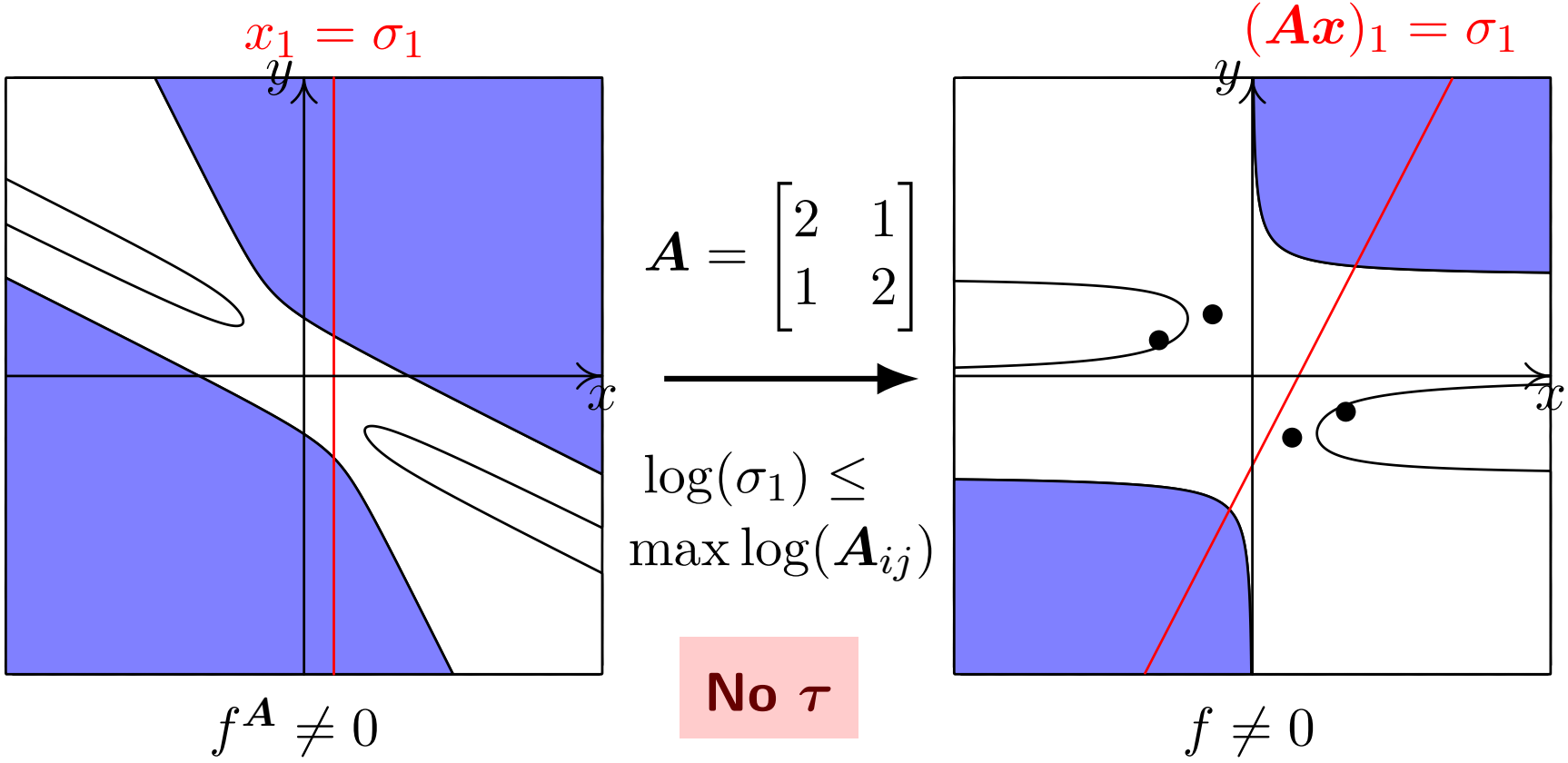

Probability of choosing σ_1 such that $V((\boldsymbol{A}\boldsymbol{x})_1 - \sigma_1, f)$ is **smooth** is $\geq 1 - \epsilon/3$

[Elliott–Giesbrecht–Schost 2020/2023]

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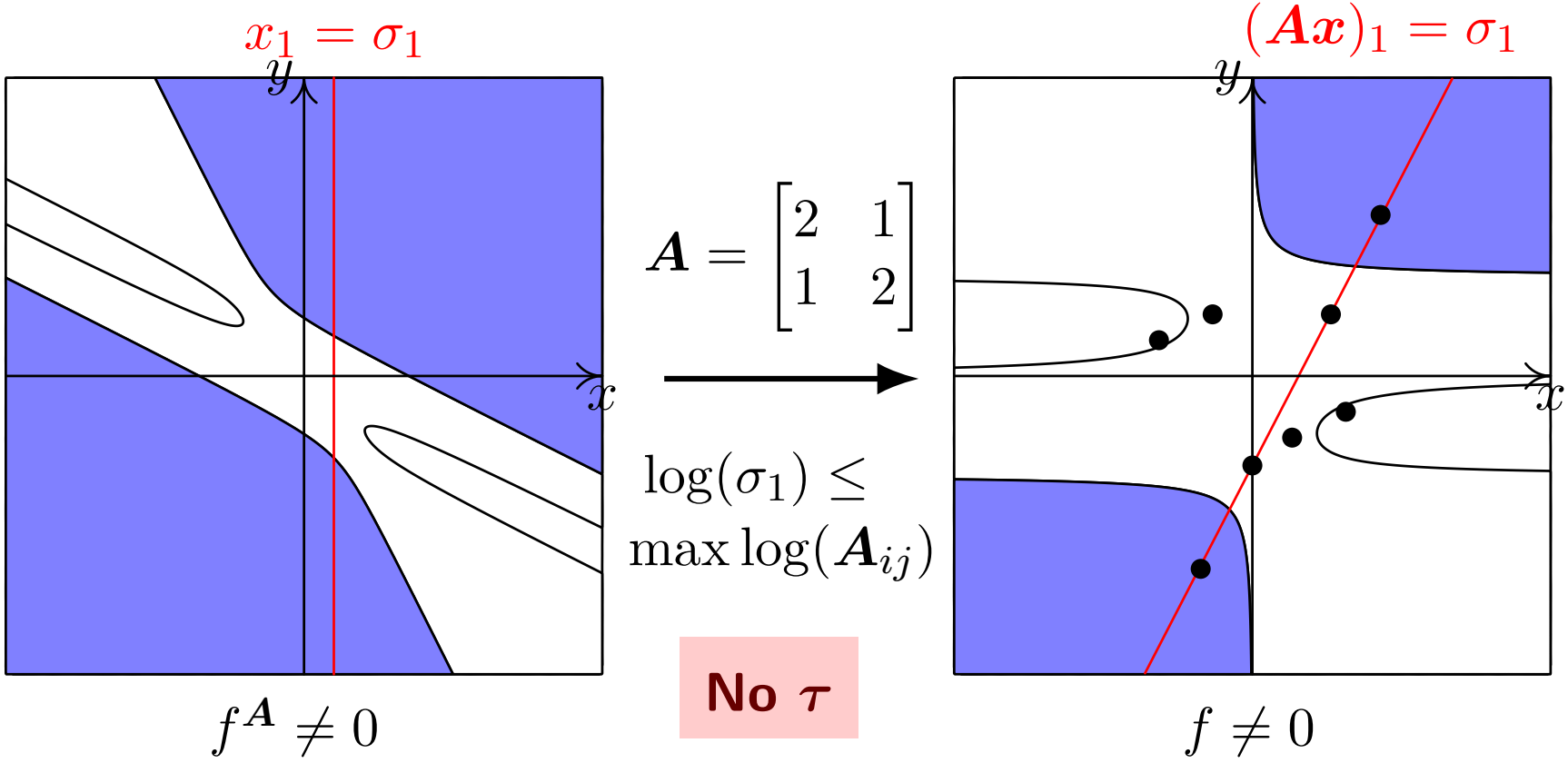

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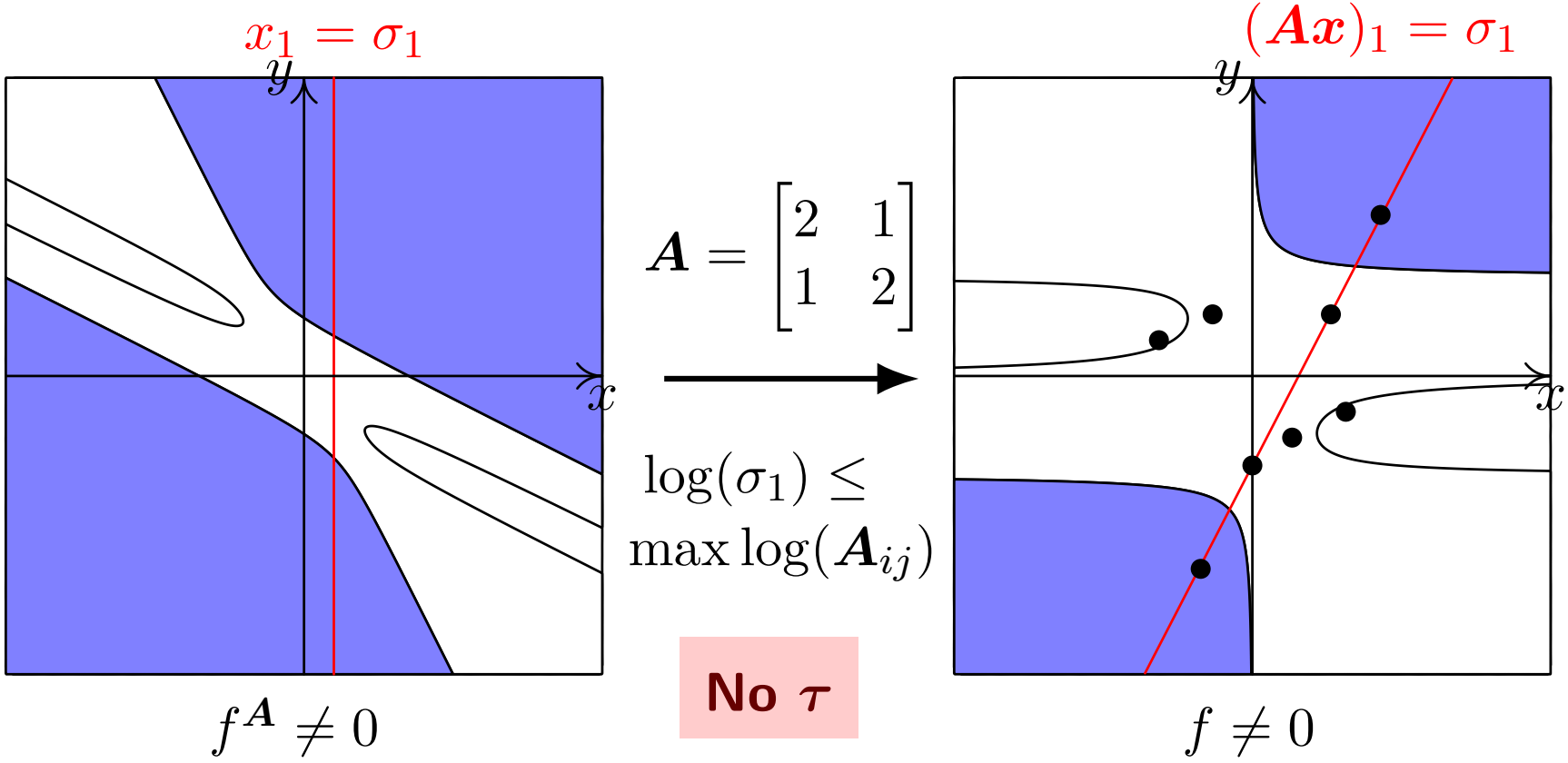

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Total bit complexity $\widetilde{O}(\tau \mathcal{P}(n, \log(1/\epsilon))\mathfrak{D}^3)$; **Probability** $\geq 1 - \epsilon$

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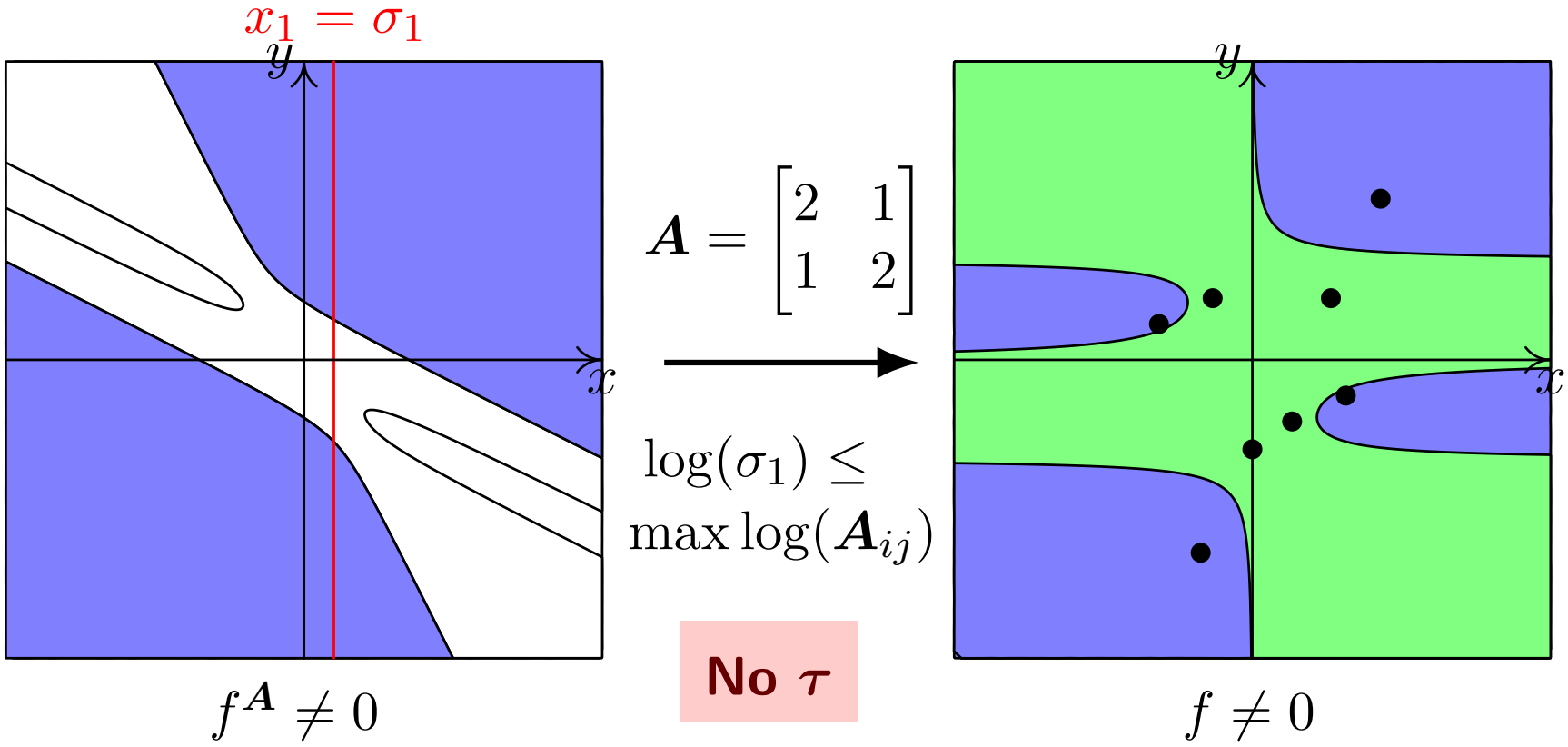

No τ

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Singular Case – Curve

No guarantee of
finiteness of solutions

$$f^{\boldsymbol{A}} = \frac{\partial f^{\boldsymbol{A}}}{\partial x_2} = \cdots = \frac{\partial f^{\boldsymbol{A}}}{\partial x_n} = 0$$

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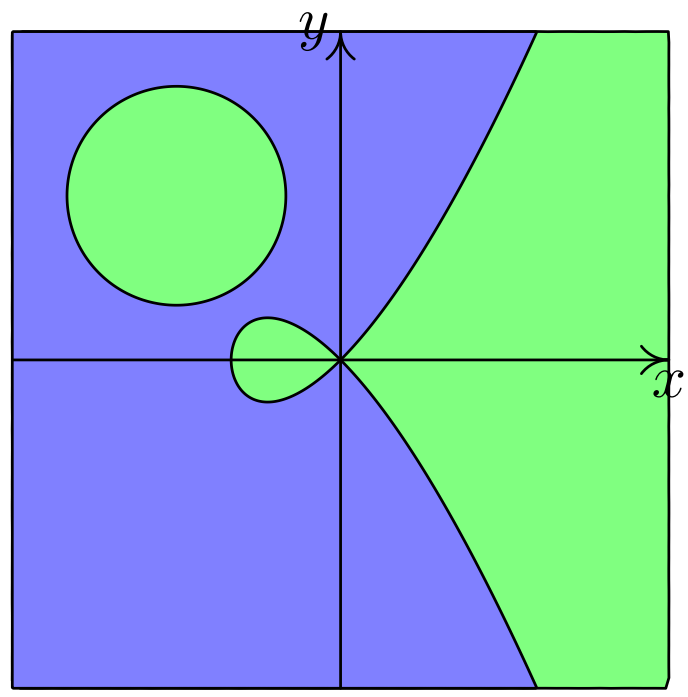

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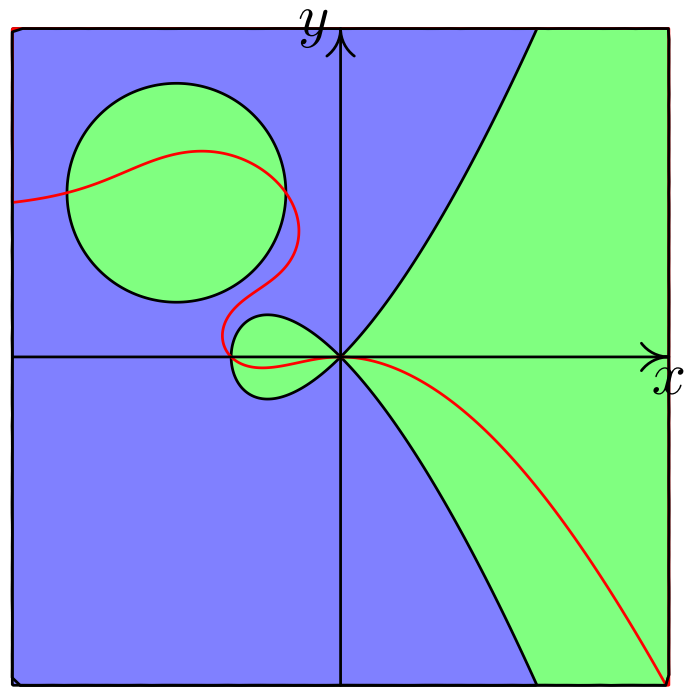

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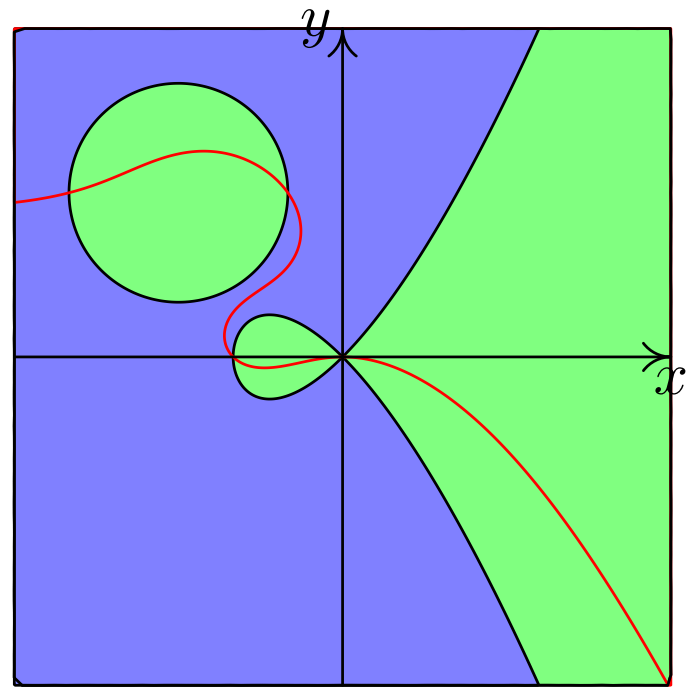
Generic $A \implies \mathcal{C}^{\boldsymbol{A}}$ is a **curve**



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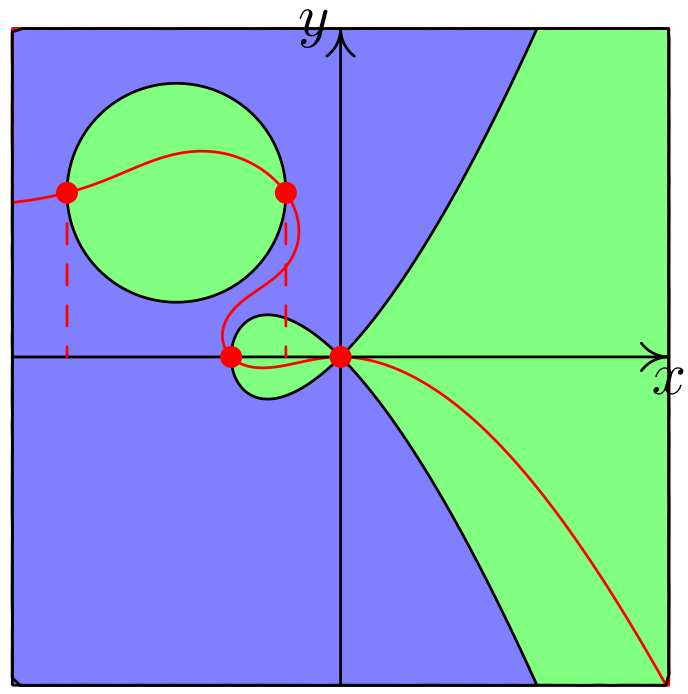
Generic A $\implies$ $\mathcal{C}^{\boldsymbol{A}}$ is a **curve**



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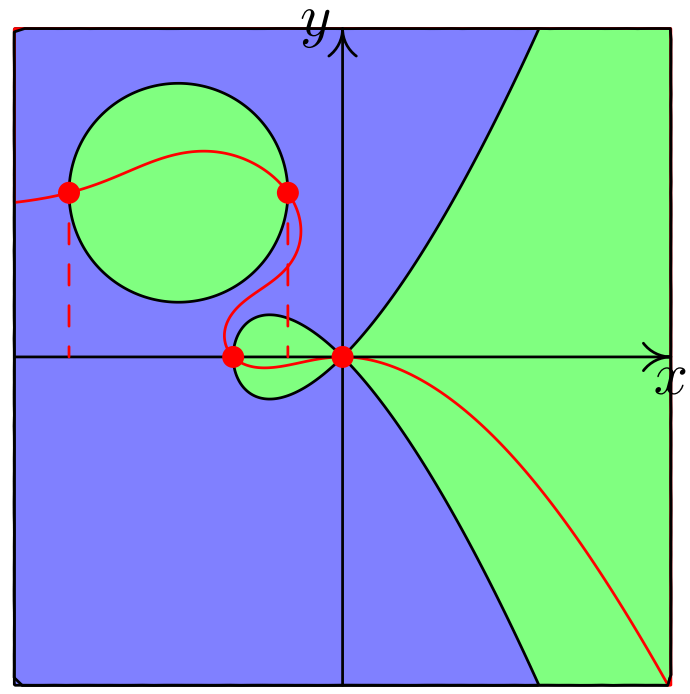

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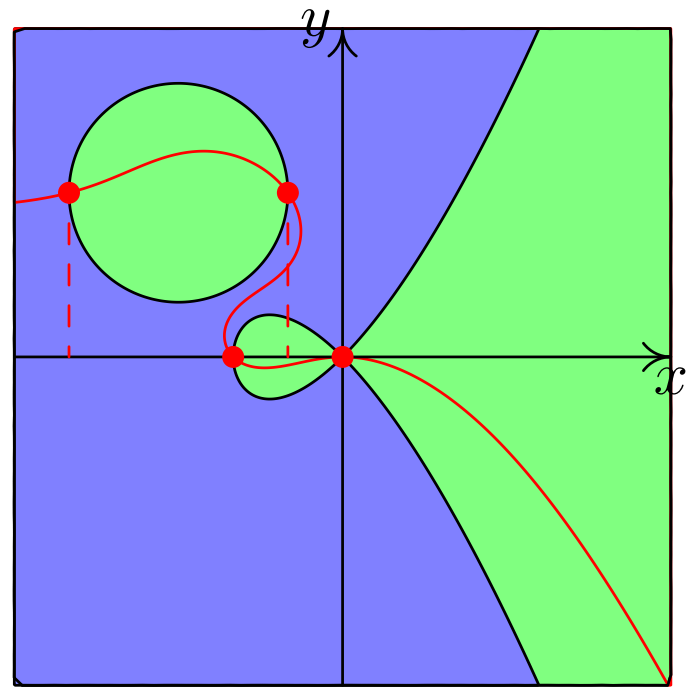

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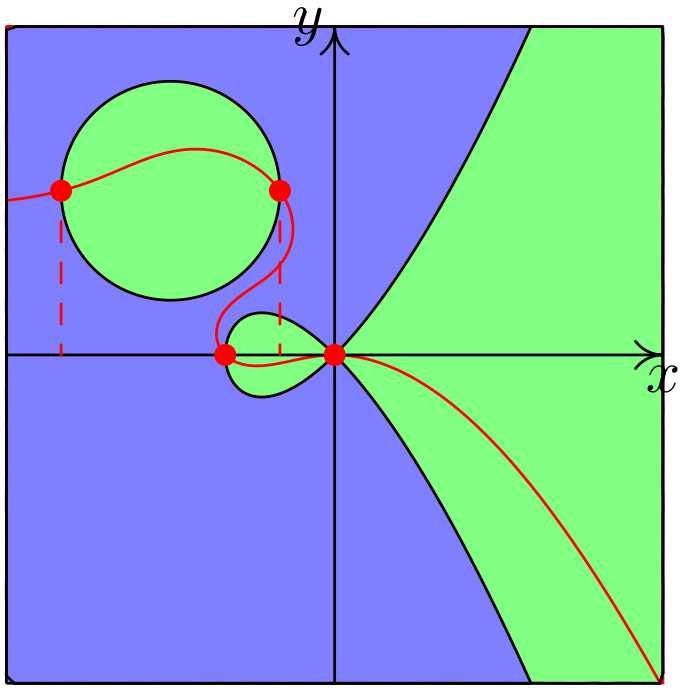
Bit cost of computing a **parametrisation** of $\mathcal{C}$:
$\widetilde{O}(\tau \mathcal{P}(n, \log(1/\epsilon))\mathfrak{D}^2)$

[Giusti–Lecerf–Salvy 2001] [Giménez–Matera 2019]

Singular Case – Intersection & Fibres



Bit cost of computing **intersection** $\mathcal{C} \cap V(f)$: $\widetilde{O}(\tau \mathcal{P}(n, \log(1/\epsilon))\mathfrak{D}^3)$

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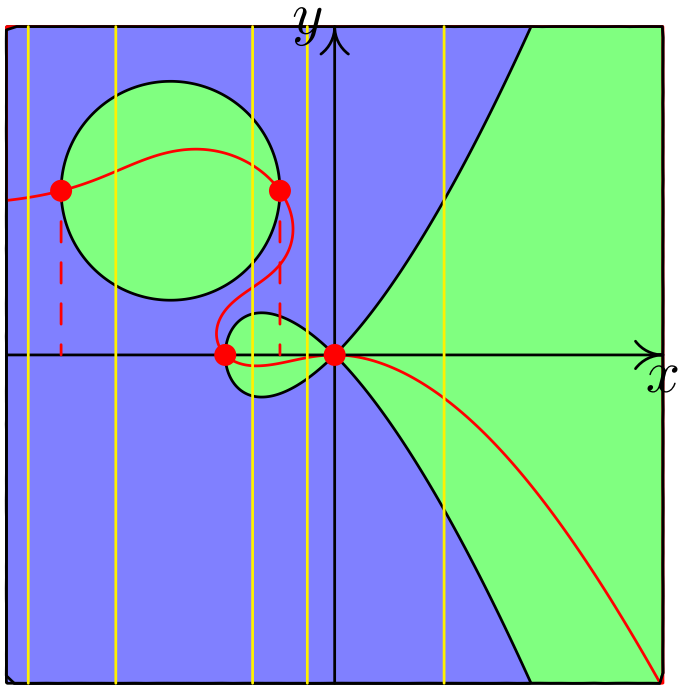

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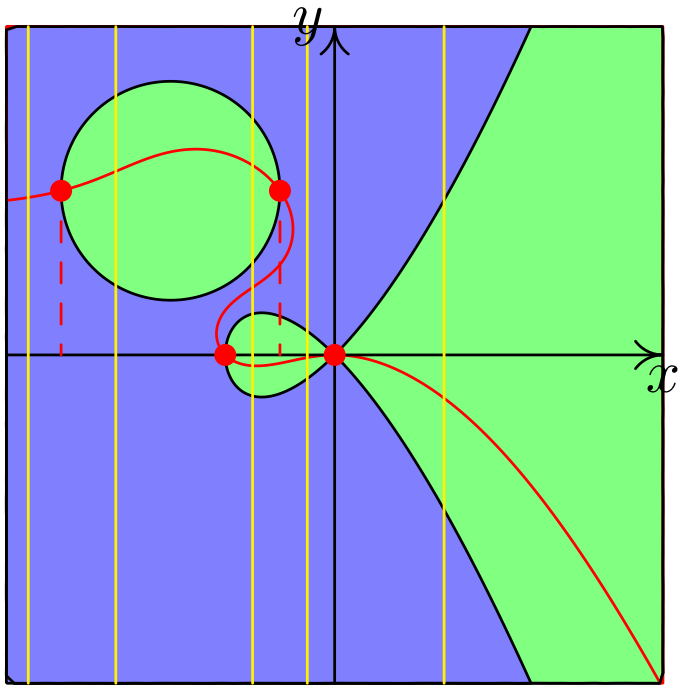

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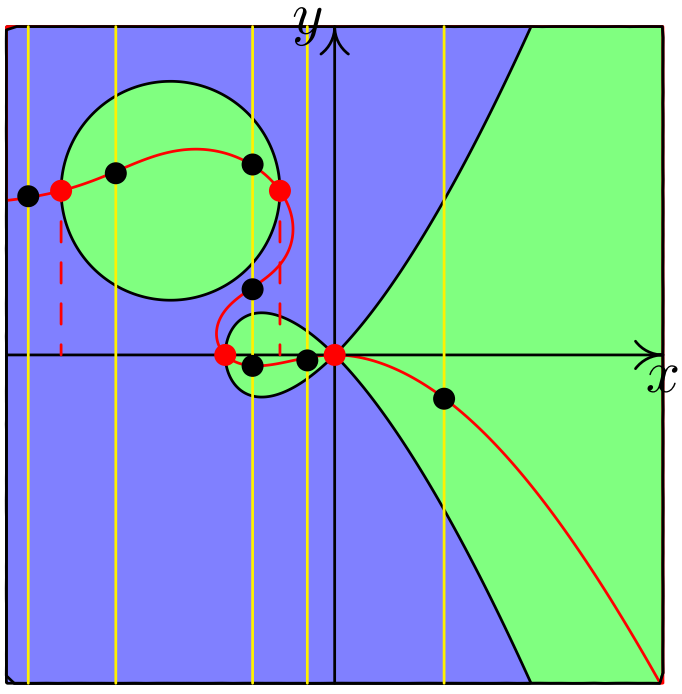
Bit cost of computing **values** between each **intersection** values: $\widetilde{O}(\tau \mathcal{P}(n, \log(1/\epsilon))\mathfrak{D}^3)$

[Sagraloff–Melhorn 2016] [Melczer–Salvy 2021]

Singular Case – Intersection & Fibres



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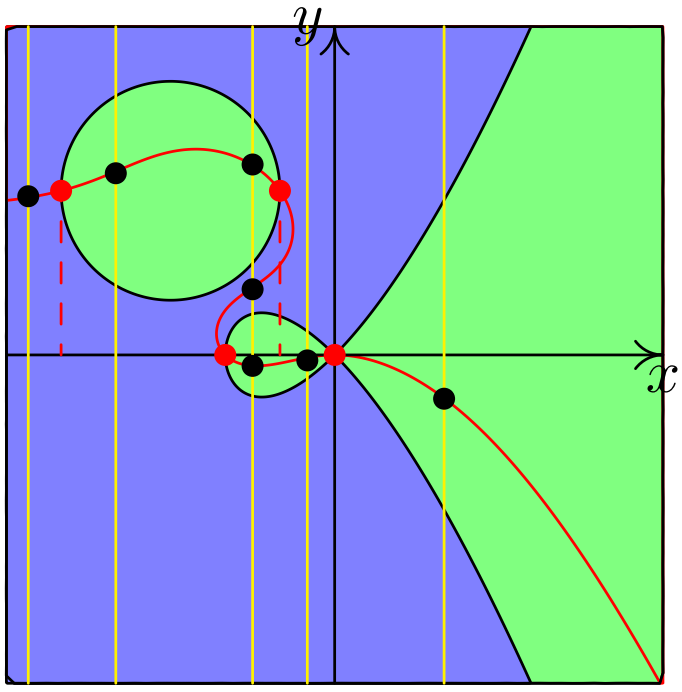

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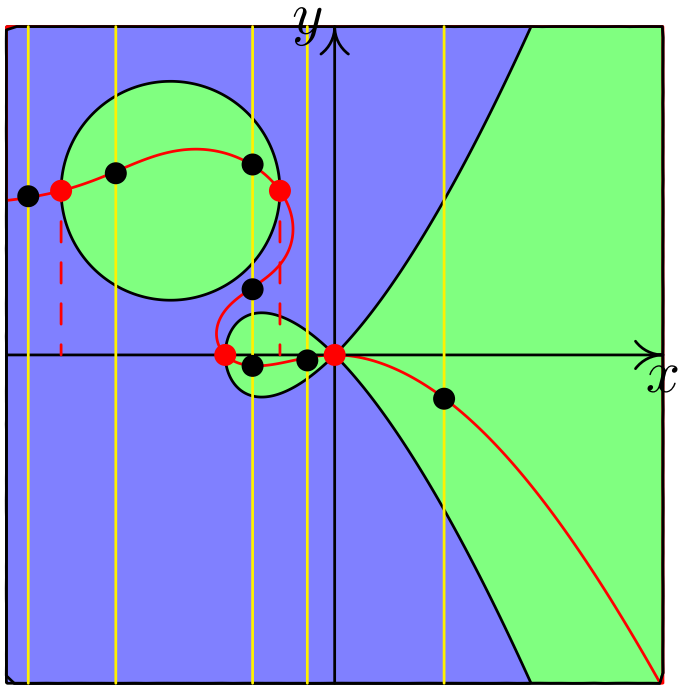
Bit cost of computing all **black** points: $\widetilde{O}(\tau \mathcal{P}(n, \log(1/\epsilon))\mathfrak{D}^5)$

Singular Case – Intersection & Fibres



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Bit cost of computing all **black** points: $\widetilde{O}(\tau \mathcal{P}(n, \log(1/\epsilon))\mathfrak{D}^5)$

Repeat with $x_1 = \sigma_1$, ... **Total** bit complexity $\widetilde{O}(\tau \mathcal{P}(n, \log(1/\epsilon))\mathfrak{D}^5)$; **Probability** of success $\geq 1 - \epsilon$

Experiments

SageMath implementation, using **msolve** for zero dimensional polynomial system solving [Berthomieu–Eder–Safey El Din 2021]

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n	CAD	[Safey El Din–Schost 2003]	New Algo
4	2min34s	1min7s	4s
5	$\gg$ 1mo	1h5min	1min7s
6	$\gg$ 1mo	2d4h	42min28s
7	$\gg$ 1mo	$\gg$ 1mo	2d10h
8	$\gg$ 1mo	$\gg$ 1mo	23d11h

Comparisons on generic dense degree 4
examples of fixed bitsize 8

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8	$\gg$ 1mo	$\gg$ 1mo	23d11h

Comparisons on generic dense degree 4 examples of fixed bitsize 8

n, k	[Safey El Din–Schost 2003]	New Algo
8, 4	OOM	$\gg$ 1d
8, 3	OOM	7h13min
8, 2	OOM	13s
12, 3	OOM	18h58min
12, 2	OOM	31s
20, 3	OOM	$\gg$ 1d
20, 2	OOM	1min40s
50, 2	OOM	23min27s
50, 1	OOM	86s
100, 0	OOM	47min39s
150, 0	OOM	8h15min

Comparisons on degree 12 examples with $n - k$ partial derivatives of degree 1

Conclusions & Further Works

New algorithms with **bit complexity** in terms of $\mathfrak{D} = \deg(f) \prod_{i=1}^{n} \deg\left(\frac{\partial f}{\partial X_i}\right)$, reflecting **structure**

Bit complexity: for $V(f)$ smooth: $\widetilde{O}(\tau \mathcal{P}(n, \log(1/\epsilon))\mathfrak{D}^3)$, otherwise $\widetilde{O}(\tau \mathcal{P}(n, \log(1/\epsilon))\mathfrak{D}^5)$

Probability of success: at least $1 - \epsilon$, $0 < \epsilon < 1$

Practical Improvements!

Long-term aims: generalise to **several** polynomials;
apply to problems from other scientific fields;
study other structures