Computing points per connected components of semi-algebraic sets

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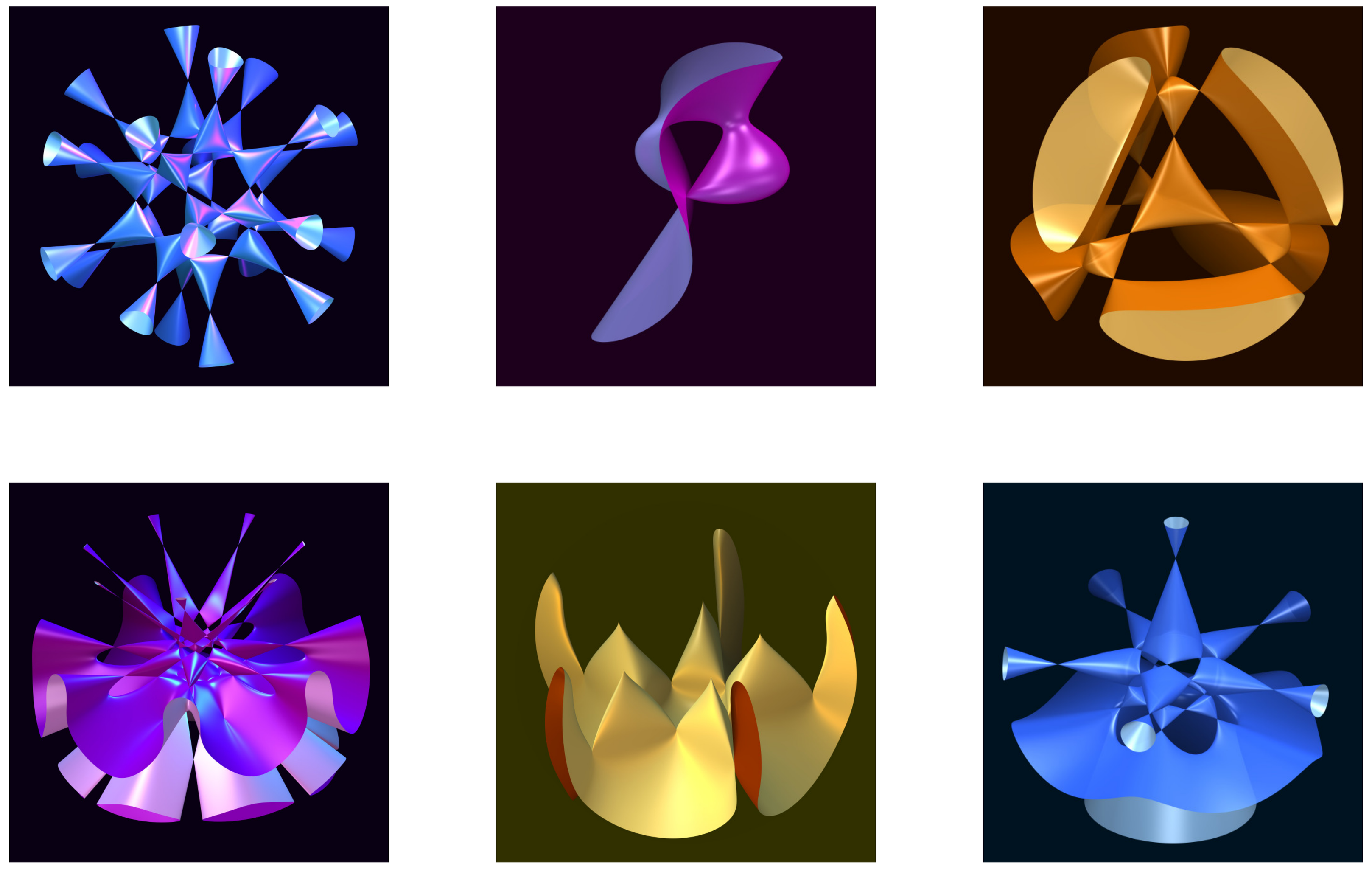
Exposition of the Problem

Semi-algebraic sets are sets of real solutions to real polynomial systems with constraints, and are generally finite unions of sets of the form

$$S \coloneqq \{\boldsymbol{x} \in \mathbb{R}^n \mid f_1(\boldsymbol{x}) = \cdots = f_r(\boldsymbol{x}) = 0, g_1(\boldsymbol{x}) > 0, \cdots, g_s(\boldsymbol{x}) > 0\}, f_i, g_j \in \mathbb{R}[\boldsymbol{X}]$$



Source: [IMAGINARY–Labs]

Deciding whether $\boldsymbol{S}$ is empty is an NP-Hard problem! [Garey–Johnson 1979].

There are always finitely many connected components of $\boldsymbol{S}$ [Whitney 1957, Łojasiewicz 1964].

The number of connected components is bounded by the Oleinik–Petrovski–Thom–Milnor bound $\boldsymbol{s^n O(d^n)}$, where $\boldsymbol{d}$ is the maximal degree of the input system [Oleinik–Petrovski 1949, Milnor 1964, Thom 1965].

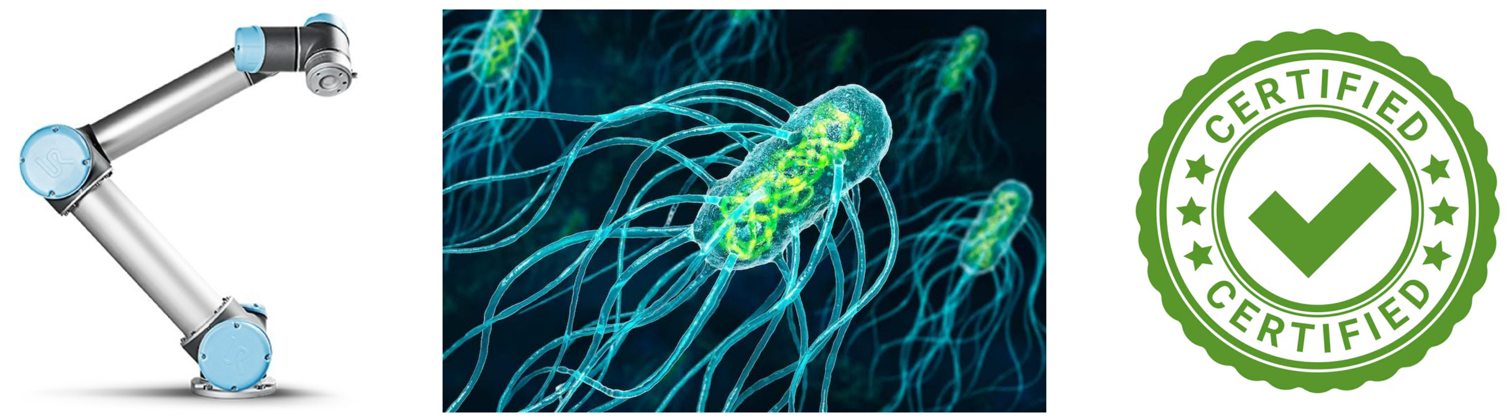
Problem Statement

Given the $f_i, g_j \in \mathbb{Q}[x_1, \dots, x_n]$ defining $\boldsymbol{S}$, compute one point per connected component of $\boldsymbol{S}$.

Applications

This problem arises in several scientific fields:

- Robotics: [Chablat–Prébet–Safey El Din–Salunkhe–Wenger 2022] [Capco–Safey El Din–Schicho 2023],
- Biology: [Feliu–Sadeghimanesh 2022] [Yabo–Safey El Din–Caillau–Gouzé 2023],
- Optimisation: [Greuet–Safey El Din 2014] [Ferguson 2022],
- Program Verification: [Goharshady–Hitarth–Mohammadi–Motwani 2023] [Prébet–Safey El Din–Schost 2024] [Maaz–Strzeboński 2025],
- Combinatorics: [Ibrahim–Salvy 2024].



The semi-algebraic sets arising from these applications have structure.

State-of-the-Art

Suppose the input system is given by $\boldsymbol{t} \coloneqq r + s$ polynomials in $\boldsymbol{n}$ variables, of maximal degree $\boldsymbol{d}$ and coefficient bitsize $\boldsymbol{\tau}$. Denote by $\boldsymbol{\mathcal{P}}$ a polynomial factor.

First algorithm: [Tarski 1951], complexity not expressible by a finite tower of exponentials (not elementary recursive).

First practical algorithm: Cylindrical Algebraic Decomposition [Collins 1975], with bit complexity $\boldsymbol{\tau d^{2^{O(n)}}}$; applicable in practice when $\boldsymbol{n} \leq 4$.

[Safey El Din–Schost 2003]
[Bank–Giusti–Heintz 2014]
[Greuet–Safey El Din 2014]

[Grigoriev–Vorobjov 1988]
[Basu–Pollack–Roy 2003]
[Elliott–Giesbrecht–Schost 2020]

Current arithmetic complexity:
$$\widetilde{O}\left(\mathcal{P}(n,t)\boldsymbol{d^{2t}(d-1)^{2n-2t+2}}\right)$$
[Le–Safey El Din 2021]

Current bit complexity:
$$\widetilde{O}\left(\mathcal{P}(\tau)\boldsymbol{d^{3(n+1)+2t+1}}\right)$$
[Elliott–Giesbrecht–Schost 2023]

These algorithms can solve more general problems, be probabilistic or make assumptions on the input, but their complexity bounds do not involve structure. However, related problems [Safey El Din–Schost 2018] [Faugère–Labahn–Safey El Din–Schost–Vu 2023] do.

Contributions

New algorithms to solve the single inequality case: $S \coloneqq \{\boldsymbol{x} \in \mathbb{R}^n \mid g(\boldsymbol{x}) > 0\}$. Structured bit complexity result, in terms of the degrees $\boldsymbol{d_i}$ of the partial derivatives of $\boldsymbol{g}$ with respect to x_i:

$$\widetilde{O}\left(\boldsymbol{\tau \mathcal{P}(n)(d d_2 \cdots d_n)^3}\right)$$

Methods used: critical points of projections, multi-homogeneous polynomial system solving, real root isolation ...

$\boldsymbol{n}$	CAD	[Elliott–Giesbrecht–Schost 2023]	New Algorithm
4	2min34s	1min7s	4s
5	$\gg$ 1mo	1h5min	1min7s
6	$\gg$ 1mo	2d4h	42min28s
7	$\gg$ 1mo	$\gg$ 1mo	2d10h
8	$\gg$ 1mo	$\gg$ 1mo	23d11h

Naive `SageMath` implementation for comparison on degree 4 examples.

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